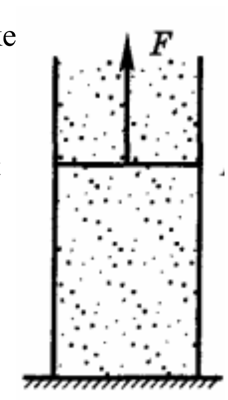
Short problems

1. Assuming the electron to be a classical particle, a sphere of radius $10^{-15}$ m and of a uniform mass density with an intrinsic angular momentum of order $\hbar$, compute the speed of rotation at the electron's equator. How does your result compare with the speed of light.

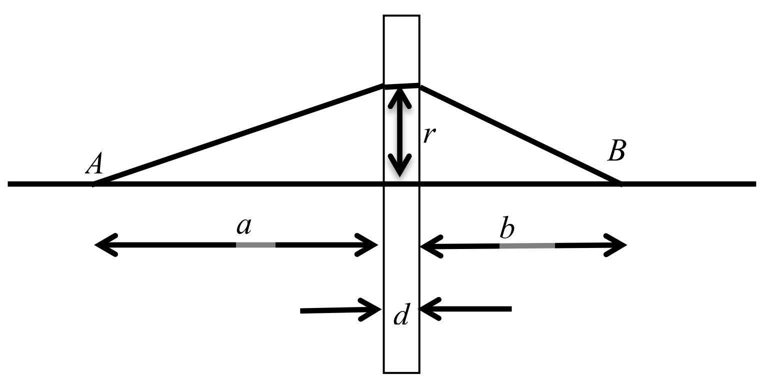
2. Two electrons can be considered distinguishable if they are well separated in space from each other, that is, their single particle wavefunctions are non- overlapping. In that case, for every possible $x_1$ value, either $\psi_\alpha(x_1)$ or $\psi_\beta(x_1)$ is zero. Show that for non-overlapping wavefunctions as defined above, the probability density for the total antisymmetric wavefunction $\psi_A^*\psi_A$ is equal to the probability density of the total symmetric wavefunction $\psi_S^*\psi_S$.

3. A particle of mass $m$ moves in a circular orbit of radius $r$ in a hypothetical atom where the force on the particle is in the form of a generalized Hooke's law: $\boldsymbol{F} = -C\boldsymbol{r}$ directed towards the center of the atom, where $C$ is the 'spring constant.' Assuming that Bohr's postulates for the atom apply in this case, in particular, that the orbital angular momentum is quantized with a quantum number $n$, derive:
   (a) The radii of the allowed orbits
   (b) The energies of these orbits in terms of the quantum number $n$. [You may take the potential energy of this "spring atom" to be $V(r) = \left(\frac{1}{2}\right) Cr^2$.]




4. A vertical cylinder contains 1.0 mole of an ideal gas at temperature $T$ under a light piston. The top of the piston is at atmospheric pressure. Find the work needed to increase the ideal gas volume by a factor of $\beta$ at $T$ = const.

5. The index of refraction of glass can be increased by diffusing in impurities. It is then possible to make a lens of constant thickness. Given a disk of radius $a$ and thickness $d$, find the radial variation of the index of refraction, $n(r)$, that will bring rays emitted from $A$ in the diagram below to a focus at $B$. Assume that the lens is thin ($d \ll a$ or $b$). Note that there are two approaches to image focusing problems. You may be more familiar with one, where you would trace light rays using Snell's law, and two rays converge to a point. This approach will lead to a very complex solution.

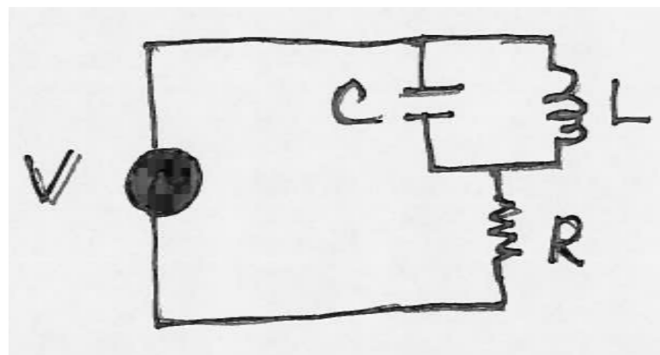



6. Consider one-particle system capable of four states ($\varepsilon_m = m\Delta$, where $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{2}$) in thermal contact with a reservoir at temperature $T$. For specific cases of $T = 0$ and $T \to \infty$, find the average energy, entropy and heat capacity.

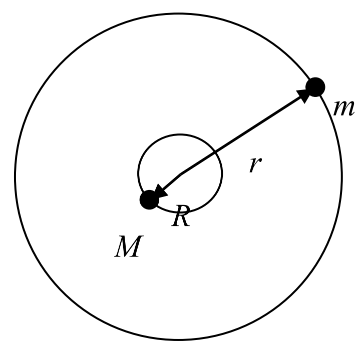
7. A lambda baryon traveling through a laboratory decays into a proton and a $\pi$ meson. The $\pi$ meson is left at rest. Find the initial laboratory kinetic energy of the lambda baryon. $m_p = 940$ MeV/c$^2$, $m_\pi = 140$ MeV/c$^2$, $m_\Lambda = 1120$ MeV/c$^2$

8. A metal can be thought of as a box filled with atoms. For each atom one electron is bound with a spring-like force (*i.e.* like Hookes' Law) such that when the electron is displaced from its equilibrium location it experiences a restoring force that is proportional to the displacement. This gives rise to a natural frequency of oscillation $\omega_0$. Now consider these atoms under the influence of an external electric field of amplitude $E_0$ and frequency $\omega$. Calculate the electric polarization of the metal as a function of frequency and $E_0$ if the density of these electrons is $n$ cm$^{-3}$.

9. In the circuit depicted on the right, the voltage source generates an AC voltage $V(t) = V_0 \cos \omega t$, where $V_0$ is fixed to a specific value while $\omega$ can be adjusted to any value. What is the power (averaged over a long time, $T \gg 1/\omega$) that is consumed by the circuit as a function of $\omega$? When $\omega \to 0$ or $\omega \to \infty$, what will happen? Is there any other case(s) when an interesting thing will happen to the power consumption?




10. A planet of mass $m$ orbiting a star of mass $M$ revolves around the center-of-mass of the system. This introduces a "reflex oscillation" of the star of amplitude $R$ (see the diagram below where the motion of the star and the planet is illustrated as seen by the observer at the center of mass of the system. $R$ and $r$ are the radii of the orbits of $m$ and $M$) and $T$ is the period of the stellar motion, viewed from outside the system. An observer searching for extrasolar planets with an astrometric telescope (measuring positions of astronomical objects with high precision) detects a stellar reflex motion of period $T = 1$ year and an amplitude $R$ that subtend an angle of $7.0 \times 10^{-9}$ radians in a star 10 light years from the earth. Assume circular orbit.




If the reflex motion is caused by an orbiting planet with Jupiter's mass ($10^{-3}$ of the solar mass), what is the planet-star distance in AU (one AU = earth-sun distance = 8.3 light minutes)?

LONG problems

1. A free particle of mass $m$ and spin $s$ is initially (at $t = 0$) in a state corresponding to the wave function

$$\psi(r) = \left(\frac{\gamma}{\pi}\right)^{3/4} e^{-\gamma r^2/2},$$

   (a) Calculate the probability density of finding the particle with momentum $\hbar \boldsymbol{k}$ at any time $t$. Is it isotropic?
   (b) What is the probability of finding the particle with energy $E$?
   (c) Examine whether the particle is in an eigenstate of the square of the angular momentum $\boldsymbol{L}$ , and of its $z$-component $L_z$, for any time $t$.

2. Consider an elastic film (2D square lattice of $N$ atoms) stretched over a rigid square with side $L$. Assume that only transverse modes, *i.e.* corresponding to displacements orthogonal to the film, can be excited and that sound velocity $u$ is frequency independent. Find:
   (a) The dispersion relationship between $\omega$ and $k$ (wave number of the sound)?
   (b) The maximum energy of an atom (often referred to as the Debye energy $\theta_D = k_B T_D$, where $T_D$ is the Debye temperature)
   (c) The energy $U$ for extreme temperatures, $T \gg T_D$ and $T \ll T_D$.
   (d) The heat capacity $C_V$, for $T \gg T_D$ and $T \ll T_D$.

3. Two identical pendulae ($\omega_0^2 = g/l$) are connected by a light coupling spring. With the coupling spring connected, one pendulum is clamped and the period of the other is found to be $T$ seconds. With neither of the connected pendulum clamped, that is, free to oscillate, what are the periods of the two normal modes?

4. Two long coaxial cylindrical metal tubes (inner radius $a$, outer radius $b$) stand vertically in a tank of dielectric oil (with dielectric constant $\varepsilon$ and mass density $\rho$) . The inner cylinder is maintained at potential $V$, while the outer one is grounded. To what height $h$ does the oil rise in the space between the tubes? For simplicity, you can assume that the top surface of the oil between the cylinders is flat and horizontal.

5. At Brookhaven National Laboratory, a beam of heavy nuclei smash into a target composed of the same, identical nuclei. The kinetic energy of the beam particles is 14.5 GeV per nucleon. What is the Lorentz factor $\gamma$ of each nucleus as seen by an observer in the center of mass frame?

   Assume that these nuclei stop each other and form one composite nuclear system, what nucleon density results? Compare this density with the average density of a neutron star with mass 1.4 solar masses and radius $R = 10$ km. Atomic nuclei have a normal density of $n_0 = 0.15$ nucleons/fm$^3$, $m_N = 1.7 \times 10^{-27}$ kg and the solar mass, $M_O = 2 \times 10^{30}$ kg.