Graduate Written Examination

Fall 2014 – Part I

University of Minnesota
School of Physics and Astronomy

Aug. 19, 2014

Examination Instructions

Part 1 of this exam consists of 10 problems of equal weight. Solve all problems.

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Constants	Symbols	Values
Speed of light in vacuum	c	3.00×10^{8} m/s
Elementary charge	e	1.60×10^{-19} C
Electron rest mass	m_e	9.11×10^{-31} kg
Electron rest mass energy	$m_e c^2$	0.511 MeV
Permeability constant	μ_o	1.26×10^{-6} H/m
Permeability constant /4π	$\mu_o/4\pi$	10^{-7} H/m
Vacuum Dielectric Constant	$1/4\pi\epsilon_0$	8.99×10^{9} N m^2/C^2
Proton rest mass	m_p	1.67×10^{-27} kg
Proton rest mass energy	$m_p c^2$	938 MeV
Neutron rest mass	m_n	1.68×10^{-27} kg
Neutron rest mass energy	$m_n c^2$	940 MeV
Planck constant	h	6.63×10^{-34} J·s
Gravitational constant	G	6.67×10^{-11} m^3/s^2·kg
Molar gas constant	R	8.31 J/mol·K
Avogadro constant	N_A	6.02×10^{23} /mol
Boltzmann constant	k_B	1.38×10^{-23} J/K
Molar volume of ideal gas at STP	V_m	2.24×10^{-2} m^3/mol
Earth radius		6.37×10^{6} m
Earth-Sun distance		1.50×10^{11} m
Stirling's Approximation		$\ln(N!) = N\ln(N) - N+$(small corrections)

Problem 1

A photon source is at the focus of a parabolic mirror and both are attached to a rocket, see Figure 1. Upon reflection the photons form a parallel beam. Find the final velocity of a rocket if it starts from rest with mass m_1 at and its final rest mass is m_2. (Be sure to use relativistic expressions throughout.)

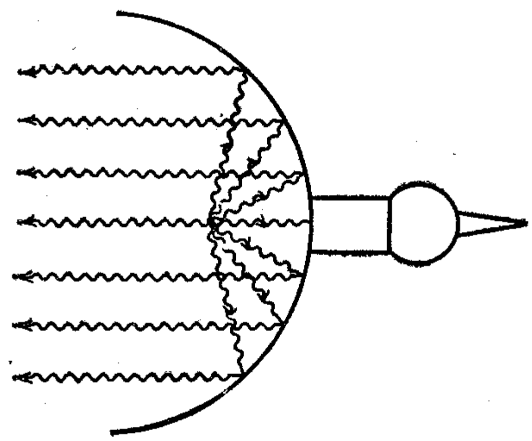

Figure 1: For Problem 2

Problem 2

In a laboratory on Earth mass m_1 is on a table and connected by a string, which runs over a frictionless pulley, to mass m_2 that is hanging from the side of the table. The table moves horizontally with acceleration a such that m_2 is tilted at a constant angle away from the table and m_1 is sliding. Find the tension in the string if the coefficient of kinetic friction between m_1 and the table is μ.

Problem 3

Consider a two-dimensional anisotropic oscillator where the potential energy is given by

$$V = (k_1 x^2 + k_2 y^2)/2.$$

1. Find the position and velocity as functions of time if the initial position and velocity are given by $r = (x_0, y_0)$ and $v = (0, 0)$.

2. What is the condition on k_1 and k_2 for the particle's trajectory to be a closed Lissajous figure?

Problem 4

The refractive index of glass can be represented approximately by the empirical relation

$$n = A + B\lambda^{-2},$$

where λ is the wavelength of light in vacuum. What are the corresponding phase and group velocities of light in glass? Do your formulae reduce to what you expect if there is no dispersion?

Problem 5

Two identical bodies, each characterized by a heat capacity at constant pressure C which is independent of temperature, are used as heat reservoirs for a heat engine. The bodies remain at constant pressure and undergo no change of phase. Initially, their temperature are T_1 and T_2, $T_1 > T_2$. At the final state, as a result of the operation of the heat engine, the bodies will attain a common final temperature T_f.

1. What is the total amount of work W done by the engine? Express the answer in terms of C, T_1, T_2, and T_f.
2. Use arguments based on entropy considerations to derive an inequality relating T_f to the initial temperatures.
3. For given initial temperatures, what is the maximum amount of work obtainable from the engine?

Problem 6

Consider a model for the hydrogen atom in which the proton is assumed to be a point charge located at the origin and the electron is described by a continuous charge distribution with spherical symmetry:

$$\rho(r) = -\frac{e}{\pi a^3} \exp\left(-\frac{2r}{a}\right)$$

where a is the Bohr radius and r is the distance from the origin. Here e is the absolute value of the charge of the electron. In the presence of an electric field $\mathbf{E}$, the proton is displaced from the origin to a new equilibrium position a distance $d \ll a$ from the origin. Calculate d and the induced dipole moment p to find an expression for the polarizability of the hydrogen atom.
Hint: use the approximation $d \ll a$ before evaluating the integral.

Problem 7

At low enough temperatures, the thermodynamic properties of a two-dimensional d-wave superconductor can be described in terms of a gas of non-interacting fermions that follow the dispersion relation $E(\mathbf{k}) = \sqrt{a^2 k_x^2 + b^2 k_y^2}$, with a, b denoting positive constants. The total number of these fermions is not conserved. Determine how the specific heat of this system depends on the temperature T in this low-temperature regime.
Hint: you do not need to evaluate the pre-factors and you can ignore the spin degeneracy.

Problem 8

The wavefunction of a particle in the ground state of a one-dimensional oscillator with potential energy $U(x) = m^2\omega^2 x^2/2$ is

$$\psi(x) = \frac{e^{-x^2/(2l^2)}}{(\pi l^2)^{1/4}} ,$$

where $l = \sqrt{\hbar/(m\omega)}$. The oscillator potential is abruptly shifted by distance a so that the potential energy becomes $U(x) = m^2\omega^2(x-a)^2/2$.

What is the probability that the particle will stay in the ground state?

Problem 9

Three sets of data are presented in Figure 2 (see next page).

1. What are the proper mathematical expressions to describe the data? Give expressions such as $X(t) = ...\ Y(t) = ...$

2. Determine all the relevant constants in the mathematical expressions.

Problem 10

The magnetic field inside a large piece of magnetic material is $\vec{B}_0$, so that $\vec{H}_0 = (1/\mu_0)\vec{B}_0 - \vec{M}$, where $\vec{M}$ is a magnetization that is frozen in the material.

1. A long narrow cylinder with the long axis parallel to $\vec{M}$ is hollowed out of the material. Find the field $\vec{B}$ at the center of the cavity in terms of $\vec{B}_0$, and $\vec{M}$. Also find $\vec{H}$ at the center of the cavity in terms of $\vec{H}_0$, and $\vec{M}$.

2. Do the same assuming that the cavity is a thin disc with the symmetry axis parallel to $\vec{M}$.

3. Do the same assuming that a small spherical cavity is hollowed out of the material.

Note: the magnetic field inside a magnetized sphere with frozen-in magnetization $\vec{M}$ is $B = \frac{2}{3}\mu_0\vec{M}$.

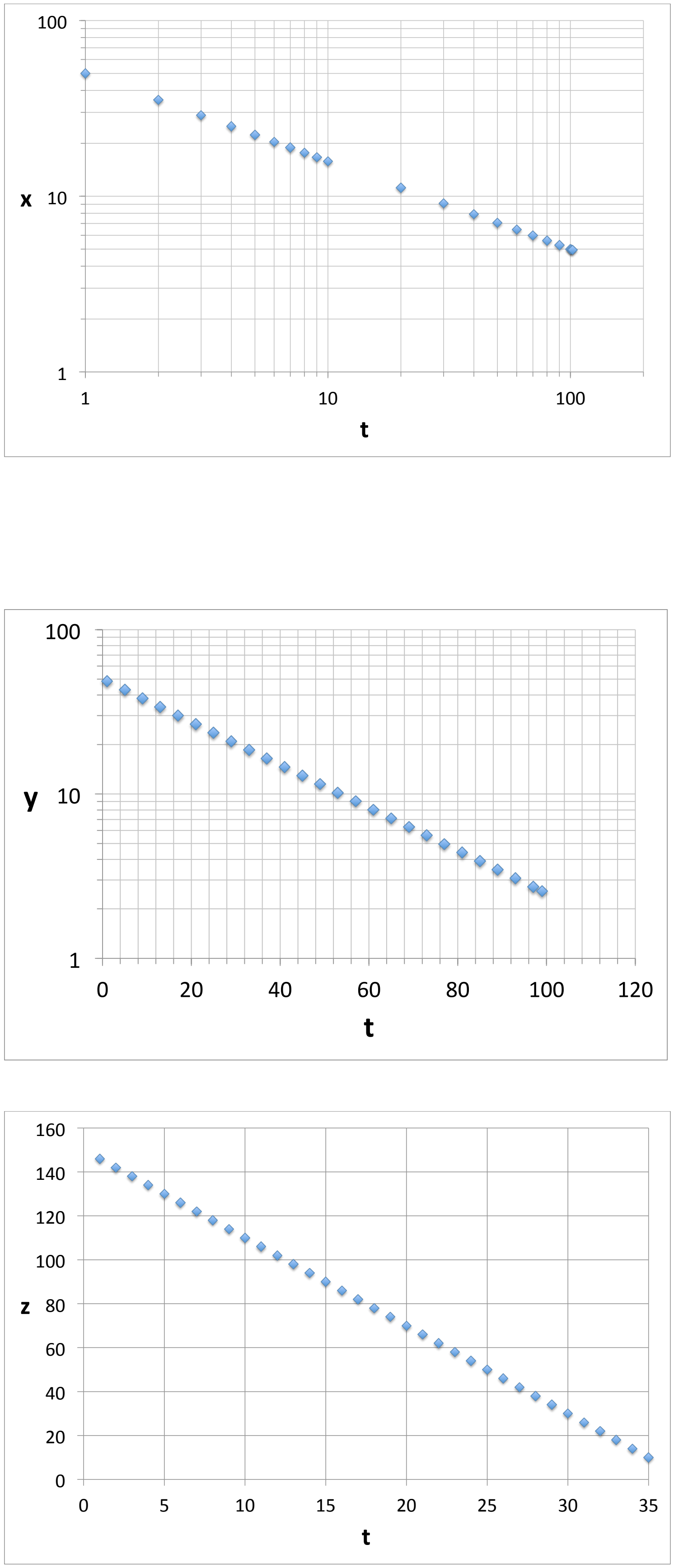

Figure 2: For Problem 9

Graduate Written Examination

Fall 2014 – Part II

University of Minnesota
School of Physics and Astronomy

Aug. 20, 2014

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The divergence of a vector field $\vec{v}$ in cylindrical coordinates is

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

1 Problem 1

A particle of mass m and electric charge q is constrained to move along a smooth vertical hoop of radius R. The hoop is in a laboratory on Earth. At the lowest point of the hoop there is another fixed charge q. Find the equilibrium positions of m and the frequency of small oscillations about the equilibrium.

2 Problem 2

Consider a long cylindrical co-axial capacitor with an inner conductor of radius a, outer conductor of radius b, and a linear dielectric in between that has permittivity $\epsilon(\rho) = \epsilon_r(\rho)\epsilon_0$, where ρ is the radial coordinate of a cylindrical coordinate system. The capacitor has a positive surface charge density σ on the inner conductor and is charged to a voltage V. You can tune the function $\epsilon(\rho)$.

A. What is the function $\epsilon(\rho)$ such that the energy density in the capacitor is independent of ρ?

B. Assuming this energy density, calculate the electric field, the electric displacement, and the polarization.

C. Calculate the bound volume and surface charges.

Note: you may need to solve part B before you can completely determine $\epsilon(\rho)$ in terms of the parameters provided. Once $\epsilon(\rho)$ is determined, use its explicit form in all other expressions (such as fields, polarization, etc).

3 Problem 3

Consider a system of N non-interacting atoms in contact with a thermal reservoir at a temperature T. Each one of these atoms can be only in one of two states: the ground state, with zero energy, and the excited state, with energy $\varepsilon > 0$.

1. Find the general expression for the free energy F of the system, and evaluate it in the limiting cases $T \to 0$ and $kT \gg \varepsilon$.

2. Compute the specific heat of the system and determine how it depends on the temperature in the cases $T \to 0$ and $kT \gg \varepsilon$.

3. The energy ε of the excited state of an atom depends on its average distance from the other atoms such that $\varepsilon = b/v^\gamma$, where b is a constant, $v = V/N$ is the volume of the system per atom, and $\gamma > 1$ is the so-called Gruneisen parameter. Find the equation of state relating the pressure P, the volume V, and the total energy E of the system.

4 Problem 4

The probability of nuclear decay when an α-particle (bound state of two protons and two neutrons) is emitted has a very strong dependence on the energy of the α-particle E. Empirically it was formulated in 1911 as the Geiger-Nuttal law:

$$\ln t_{1/2} = a_1 \frac{Z}{\sqrt{E}} - a_2 \,, \tag{1}$$

where $t_{1/2}$ is the half-life (in seconds), Z is the electric charge of the final nucleus, and $a_{1,2}$ are constants.
The law was explained in 1928 by George Gamow who used the WKB approximation to calculate the transmission coefficient for the potential

$$U(r) = \begin{cases} -U_0 & \text{at } r < r_0 \\ Z\alpha/r & \text{at } r > r_0, \end{cases} \tag{2}$$

which represents a potential well at $r < r_0$ and a Coulomb repulsion at $r > r_0$.

Use this potential to find the coefficient a_1 for $E \ll Z\alpha/r_0$, where you can neglect the size of the nucleus, i.e. take the limit $r_0 \to 0$. Determine the numerical value of a_1 when the energy E is measured in MeV.

Reminder: the WKB approximation consists of assuming that the phase of the wave function in a weakly inhomogeneous potential $U(x)$ can be written as $\int k(x)\,dx$ where $k(x)$ is the wave number corresponding to the value of the potential at point x.
Hint: an integral of the form $\int_0^1 \left(\sqrt{\frac{1}{x} - 1}\right) dx$ can be solved by substituting $x = \sin^2 \phi$.

5 Problem 5

A slow neutron hits a hydrogen atom in its ground state and forms a final state containing a deuterium atom and a photon,

$$n + H \to D + \gamma \,.$$

The nucleus of the deuterium atom, a deuteron, is the bound state of a neutron and proton with a binding energy of 2.23 MeV.

What is the velocity of the deuterium in the final state in the limit of zero kinetic energy of the initial neutron? In this limit, what is the total probability to find the electron in any excited state of the deuterium atom?

Hints: (1) you can treat the deuterium motion as non-relativistic, $v << c$;
(2) transition to a coordinate system moving with velocity $\vec{v}$ leads to multiplication of the wave function $\psi(\vec{r})$ by a factor $\exp[-im\vec{v}\,\vec{r}/\hbar]$;
(3) the hydrogen wave function for the ground state is $\psi_0 = (1/\sqrt{\pi a^3})\exp(-r/a)$, where a is the Bohr radius of the atom: $a = \hbar/(mc\alpha)$ with $\alpha = 1/137$.