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The relationship between noise correlation and the Green's function in the presence of degeneracy and the absence of equipartition

Victor C. Tsai

Geologic Hazards Science Center, United States Geological Survey, Golden, CO 80401, *USA. E-mail: vtsai@post.harvard.edu*

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SUMMARY

Recent derivations have shown that when noise in a physical system has its energy equipartitioned into the modes of the system, there is a convenient relationship between the cross correlation of time-series recorded at two points and the Green's function of the system. Here, we show that even when energy is not fully equipartitioned and modes are allowed to be degenerate, a similar (though less general) property holds for equations with wave equation structure. This property can be used to understand why certain seismic noise correlation measurements are successful despite known degeneracy and lack of equipartition on the Earth.

Key words: Surface waves and free oscillations; Seismic tomography; Theoretical seismology; Statistical seismology; Wave propagation.

1 INTRODUCTION

Beginning with the work of Lobkis & Weaver (2001), a number of authors have shown that under very special circumstances, there exists a convenient relationship between the cross correlation of time-series recorded at two points and the Green's function of the system. The assumptions made in the various derivations ranges substantially, with some authors assuming that the energy in normal modes is uncorrelated and perfectly equipartitioned (Lobkis & Weaver 2001), some assuming an isotropic energy flux (Snieder 2004; Sanchez-Sesma & Campillo 2006; Sanchez-Sesma *et al.* 2006; Tsai 2009), and others relying on reciprocity relations (Wapenaar 2004; Wapenaar *et al.* 2006).

To make use of these properties, it is further often assumed that certain systems, like the Earth, satisfy all of the assumptions of the theories. However, some assumptions are clearly not satisfied, bringing into question the applicability of the theories. For example, it is well known that most ambient seismic noise is generated at the surface and that modes with higher sensitivity near the surface will be preferentially excited, resulting in a lack of equipartition; it is also known that microseisms are primarily generated in certain oceanic regions and therefore do not provide an isotropic incidence of energy flux. Do relationships between the cross correlation and Green's function still exist under less restrictive conditions? The relationship between the assumptions is also not entirely clear. For example, are the assumptions of equipartition and that of isotropic energy flux equivalent? In this work, we address some of these questions. Using a normal mode approach, we find important differences in the cross correlation–Green's function relationship when modes are equipartitioned or not and when modes are degenerate or not. We then show a few simple examples of applying these relationships.

2 DEFINITIONS

The approach taken in this work is to consider a mathematical system in which wave-like modes exist [as in Lobkis & Weaver (2001)] and determine the consequences of the system satisfying various assumptions. Specifically, we assume that the system of interest can be written in the form

$$\frac{\partial^2 u(\boldsymbol{x}, t)}{\partial t^2} = \mathcal{L}[u(\boldsymbol{x}, t)], \tag{1}$$

where $\boldsymbol{x}$ is a spatial variable, t is time and $\mathcal{L}$ is a spatial operator with given boundary conditions such that there is a basis of spatial eigenfunctions (normal modes), $s_k(\boldsymbol{x})$. That is, we assume that

$$\mathcal{L}[s_k(\boldsymbol{x})] = \lambda_k s_k(\boldsymbol{x}) = -\omega_k^2 s_k(\boldsymbol{x}), \tag{2}$$

where the eigenfrequencies w_k are real, $s_k(\boldsymbol{x})$ are orthonormal, satisfying $\langle s_k, s_{k'} \rangle = \delta_{kk'}$, and any arbitrary function $f(\boldsymbol{x})$ in the solution space can be expressed as

$$f(\boldsymbol{x}) = \sum_k A_k s_k(\boldsymbol{x}). \tag{3}$$

In the above, $\langle \cdot\,, \cdot \rangle$ denotes the appropriate inner product, δ_{ij} is the Kronecker delta, and the sum on k may (or may not) be infinite. One should note that the acoustic wave equation and elastic wave equation satisfy these assumptions, regardless of the number of spatial dimensions. One may also note that these two equations do not include attenuation.

Considering the previous assumptions, standard separation of variables then yields the general solution

$$u(\boldsymbol{x}, t) = \sum_k a_k s_k(\boldsymbol{x}) \cos(\omega_k t + \phi_k), \tag{4}$$

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where a_k and ϕ_k are constants determined by the initial conditions. We define two modes l and m to be equipartitioned if $a_l = a_m$.

The Green's function, $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$, for an impulse at location $\boldsymbol{x}_0$ and time t_0, satisfies

$$\frac{\partial^2 G}{\partial t^2} - \mathcal{L}[G] = \delta(\boldsymbol{x} - \boldsymbol{x_0})\delta(t - t_0), \tag{5}$$

where $\delta(x)$ is the Dirac delta function. Choosing $t_0 = 0$ for convenience, G can be written as a modal sum as in eq. (4), with coefficients $a_k = s_k(\boldsymbol{x}_0)/\omega_k$, $\phi_k = -\pi/2$ when $t > 0$. Writing this out explicitly yields

$$G(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) = \begin{cases} \sum_k \frac{1}{\omega_k} s_k(\boldsymbol{x}) s_k(\boldsymbol{x_0}) \sin \omega_k t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}. \tag{6}$$

We further define the extended Green's function as

$$\begin{aligned} G^{Ex}(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) &\equiv G(\boldsymbol{x}, t; \boldsymbol{x_0}, 0) - G(\boldsymbol{x}, -t; \boldsymbol{x_0}, 0) \\ &= \sum_k \frac{1}{\omega_k} s_k(\boldsymbol{x}) s_k(\boldsymbol{x_0}) \sin \omega_k t, \end{aligned} \tag{7}$$

for all t. The time derivative is then given by

$$\frac{\mathrm{d}G^{Ex}}{\mathrm{d}t} = \sum_k s_k(\boldsymbol{x}) s_k(\boldsymbol{x_0}) \cos \omega_k t. \tag{8}$$

Finally, we define the normalized cross correlation as

$$C^T_{fg}(t) \equiv \frac{1}{2T} \int_{-T}^{T} f(\tau) g(\tau + t) \mathrm{d}\tau, \tag{9}$$

and the limiting value as

$$C_{fg}(t) \equiv f \star g(t) \equiv \lim_{T \to \infty} C^T_{fg}(t). \tag{10}$$

For this choice, cross correlations between (infinitely long) sinusoidal signals are well defined. An important example of this, which is used throughout this work, is when $f(t) = \cos(\omega_k t + \phi_k)$ and $g(t) = \cos(\omega_{k'} t + \phi_{k'})$. For this choice of f and g, direct calculation yields

$$C_{fg}(t) = \begin{cases} \frac{1}{2} \cos(\omega_k t + \phi_{k'} - \phi_k) & \text{if } \omega_k = \omega_{k'} \\ 0 & \text{if } \omega_k \neq \omega_{k'} \end{cases}. \tag{11}$$

In the following sections, the subscript 'fg' on cross correlations will refer to the locations of the recorded time-series (rather than functions).

3 RELATIONSHIPS BETWEEN THE CROSS CORRELATION AND THE GREEN'S FUNCTION

We now provide the relationship between cross correlations and the Green's function under various assumptions.

3.1 Equipartitioned, non-degenerate modes

In this most restricted case, it is assumed that energy is equipartitioned between all modes so that all modes have equal modal amplitudes, $a_k = \text{const.} = A$, so that $u(\boldsymbol{x}, t)$ can be written as

$$u(\boldsymbol{x}, t) = A \sum_k s_k(\boldsymbol{x}) \cos(\omega_k t + \phi_k). \tag{12}$$

It is also assumed that no modes are degenerate, that is, $w_k \neq w_{k'}$ when $k \neq k'$. With these assumptions, applying eq. (11) term by term yields

$$C_{x_1 x_2}(t) = \frac{A^2}{2} \sum_k s_k(\boldsymbol{x_1}) s_k(\boldsymbol{x_2}) \cos \omega_k t. \tag{13}$$

Comparing eq. (13) with eq. (8), one immediately finds that

$$C_{x_1 x_2}(t) = \frac{A^2}{2} \frac{\mathrm{d}G^{Ex}(\boldsymbol{x_1}, t; \boldsymbol{x_2}, 0)}{\mathrm{d}t}, \tag{14}$$

that is, the cross correlation is identical to the time derivative of the Green's function, up to an amplitude factor. Expressions similar to eq. (14) have been previously noted by a number of authors including Lobkis & Weaver (2001), Wapenaar (2004) and Roux *et al.* (2005), though not with the same assumptions. Importantly, this relationship holds for a completely deterministic process in which the phases, ϕ_k, are arbitrary but unchanging and no assumption regarding the randomness of these phases through time is required.

3.2 Equipartitioned, degenerate modes

Degeneracy of modes, with $\omega_k = \omega_{k'}$ for some $k \neq k'$, is common in many systems. For example, symmetric or periodic boundary conditions often leads to degeneracy. When modes are allowed to be degenerate, eq. (13) no longer holds because of the existence of cross terms between degenerate modes. For these terms to disappear, one must make the additional assumption [as in Lobkis & Weaver (2001)] that there is an ensemble of states in which the phases, ϕ_k, are randomly chosen in each realization or, equivalently, that the ϕ_k are periodically reset to random values. To demonstrate the result of this additional assumption, we take the ensemble point of view, assuming there exist N different realizations of eq. (12) with randomly chosen ϕ_k so that

$$u_j(\boldsymbol{x}, t) = A \sum_k s_k(\boldsymbol{x}) \cos(\omega_k t + \phi_{jk}), \tag{15}$$

where ϕ_{jk} are randomly chosen, $1 \leq j \leq N$. The ensemble cross correlation C^E_{fg} is set equal to the average of the cross correlations for these N realizations. Assuming that modes l and m are degenerate ($\omega_l = \omega_m$), then the part of $C^E_{x_1 x_2}$ due to just those terms would be

$$\begin{aligned} C^{Elm}_{x_1 x_2} = \frac{A^2}{2} \Big[& s_l(\boldsymbol{x_1}) s_l(\boldsymbol{x_2}) \cos(\omega_l t) + s_m(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos(\omega_m t) \\ & + \frac{1}{N} s_l(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos(\omega_l t + \phi_{1m} - \phi_{1l}) \\ & + \frac{1}{N} s_l(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos(\omega_l t + \phi_{2m} - \phi_{2l}) + \ldots \Big]. \end{aligned} \tag{16}$$

If $\phi_{jm} - \phi_{jl}$ are randomly distributed (i.e. uncorrelated), then the sum of the cross terms will grow as $\sqrt{N}$, so that eq. (16) reduces to

$$\begin{aligned} C^{Elm}_{x_1 x_2} = \frac{A^2}{2} \Big[& s_l(\boldsymbol{x_1}) s_l(\boldsymbol{x_2}) \cos(\omega_l t) + s_m(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos(\omega_m t) \\ & + \frac{1}{\sqrt{N}} s_l(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos\left(\omega_l t + \phi^{\text{avg}}_{lm}\right) + \ldots \Big], \end{aligned} \tag{17}$$

where ϕ^{avg}_{lm} is the average of $\phi_{jm} - \phi_{jl}$ (which can be calculated by summing phases in the complex plane). As $N \to \infty$, then again the cross terms become vanishingly small

$$\begin{aligned} \lim_{N \to \infty} C^{Elm}_{x_1 x_2} = \frac{A^2}{2} [& s_l(\boldsymbol{x_1}) s_l(\boldsymbol{x_2}) \cos(\omega_l t) \\ & + s_m(\boldsymbol{x_1}) s_m(\boldsymbol{x_2}) \cos(\omega_m t)]. \end{aligned} \tag{18}$$

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Since the same argument can be made for all pairs of degenerate modes, we retrieve the relationship

$$\lim_{N\to\infty} C^E_{x_1x_2}(t) = \frac{A^2}{2}\frac{\mathrm{d}G^{Ex}(\boldsymbol{x_1}, t; \boldsymbol{x_2}, 0)}{\mathrm{d}t}. \tag{19}$$

Thus, we observe that only when ϕ_{jk} samples sufficiently many states does the cross correlation converge upon the Green's function. This result is similar to that of Lobkis & Weaver (2001).

For a system with attenuation, the ϕ_k [e.g. of eq. (12)] are naturally reset to random values after an e-folding attenuation time as long as there exists random forcing through time (since the effects of forcing from times sufficiently far in the past are attenuated away). This is an example of a case in which the periodic resetting of phases results in an effective sampling of different ensemble states. In such a case, the number of realizations, N, would roughly be equivalent to the number of e-folding attenuation times in the time-series used. Provided that eq. (11) is well approximated within the e-folding time (i.e. attenuation is weak), then we again achieve eq. (19) as a result.

3.3 Non-equipartitioned, non-degenerate modes

We now relax the assumption of equipartition but consider the case where there are no degenerate modes. Again applying eq. (11) term by term yields

$$C_{x_1x_2}(t) = \sum_k \frac{a_k^2}{2} s_k(\boldsymbol{x_1}) s_k(\boldsymbol{x_2}) \cos\omega_k t. \tag{20}$$

Comparing eq. (20) with eq. (8), we observe that while the summed expressions cannot be compared directly, each term is still equivalent up to an amplitude factor, $a_k^2/2$, which can be different for each k. In many applications, including the seismological one, 'phase' or 'traveltime' information is useful regardless of whether amplitude measurements can be made easily (whether due to instrumental limitations or structural complexities). (It is perhaps useful to note that if one only has band-limited data, the full Green's function can never be reconstructed anyway.) Thus, we observe that the cross correlation still yields a useful portion of the Green's function. Specifically, if one filters both the cross correlation and the Green's function at frequency ω_k, then one is left with only the kth term of the sum, hence

$$\mathcal{F}_{\omega_k}[C_{x_1x_2}(t)] = A_k \mathcal{F}_{\omega_k}\left[\frac{\mathrm{d}G^{Ex}(\boldsymbol{x_1}, t; \boldsymbol{x_2}, 0)}{\mathrm{d}t}\right], \tag{21}$$

where $\mathcal{F}_\omega$ is a perfect filter at frequency ω and $A_k = a_k^2/2$. A similar relationship was previously drawn by Sanchez-Sesma & Campillo (2006) for the less general case of propagating *P, SV* and *SH* waves.

3.4 Non-equipartitioned, degenerate modes

Finally, we relax both the assumption of equipartition and of non-degeneracy. Following the procedure of Section 3.2, we can still apply eq. (11) and obtain

$$\lim_{N\to\infty} C^E_{x_1x_2}(t) = \sum_k \frac{a_k^2}{2} s_k(\boldsymbol{x_1}) s_k(\boldsymbol{x_2}) \cos\omega_k t. \tag{22}$$

As in Section 3.3, there is term-by-term equality of the cross correlation and the Green's function. However, if degenerate modes have different relative amplitudes (i.e. $a_l \neq a_m$ when $\omega_l = \omega_m$), then even a filtered response will not yield results equivalent up to an amplitude scale factor.

This analysis shows that the necessary condition on (perfect) retrieval of Green's function 'phase' information using cross correlations is that all degenerate modes are equipartitioned. The lack of equipartition between non-degenerate modes only results in a loss of amplitude information. When this equipartition of degenerate modes exists, then

$$\mathcal{F}_{\omega_k}\left[\lim_{N\to\infty} C^E_{x_1x_2}(t)\right] = A_k \mathcal{F}_{\omega_k}\left[\frac{\mathrm{d}G^{Ex}(\boldsymbol{x_1}, t; \boldsymbol{x_2}, 0)}{\mathrm{d}t}\right], \tag{23}$$

where the $A_k = a_k^2/2$ are the same for each degeneracy group.

It may further be observed that if a subset of these degenerate modes, s_l (with $l \in S$), are equipartitioned with amplitude a_l, then it may be possible to partially retrieve Green's function 'phase' information. This would be possible, for example, if the arrival phases of all other degenerate modes $s_m (m \notin S)$ are cleanly separated in time from modes s_l such that a time windowing of both sides of eq. (23) results in picking out only modes s_l. Since windowing introduces neighbouring frequencies, this partial retrieval would inherently make use of approximately degenerate modes (and their group arrivals). This case of partially equipartitioned modes may apply in many cases. For example, on the Earth, a subset of fundamental-mode Rayleigh waves may be approximately equipartitioned while all other modes (including higher order modes) may not. On the Earth, higher order modes are typically poorly excited (most forcing is near-surface and excites fundamental-mode waves more easily), resulting in much lower modal amplitudes (a_k) compared to the fundamental-mode amplitudes. This can therefore help explain why phase information for fundamental-mode Rayleigh waves is typically easily obtained, but similar phase information for higher order (body wave) modes is difficult to obtain.

4 EXAMPLES

In this section, three examples are given of systems in which the results of Section 3 can be applied. In the first example, the simple case of waves on a periodic string, many of the properties discussed in Section 3 can be easily understood. Prior to discussing the third example of waves on the Earth, we provide a second example of waves on a periodic square. This example has many similarities to the Earth case and has results that can be generalized to the Earth, yet is simpler to understand.

4.1 Waves on a periodic string of length L

A string of length L with periodic boundary conditions is an example of a simple system in which waves have degeneracy. In this case, $\mathcal{L} = c^2\partial^2/\partial x^2$, and for each $\omega_k = k\pi c/L$ (where k is an integer, as before) there are two eigenfunctions $s_{k1}(x) = \sin(k\pi x/L)$ and $s_{k2}(x) = \cos(k\pi x/L)$ for $1 \leq k < \infty$, each with two initial conditions. By the results of Section 3.4, we know that if each pair of degenerate modes has equal amplitudes, then eq. (23) holds.

Equivalently, one can express these four (standing wave) modes as four travelling waves, two of which travel in the positive x direction, two of which travel in the negative direction. Written as travelling waves

$$\begin{aligned} &a_{k1}\sin(\kappa x)\cos(\omega t + \phi_{k1}) + a_{k2}\cos(\kappa x)\cos(\omega t + \phi_{k2}) \\ &\quad = \left[-\frac{a_{k1}}{2}\sin(\omega t + \phi_{k1} - \kappa x) + \frac{a_{k2}}{2}\cos(\omega t + \phi_{k2} - \kappa x)\right] \\ &\quad + \left[\frac{a_{k1}}{2}\sin(\omega t + \phi_{k1} + \kappa x) + \frac{a_{k2}}{2}\cos(\omega t + \phi_{k2} + \kappa x)\right]. \end{aligned} \tag{24}$$

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If, now, one assumes that $-a_{k1} + a_{k2}$ and $a_{k1} + a_{k2}$ are equal averaged over an ensemble (as in Section 3.2, corresponding to equal contributions from waves propagating in the positive and negative directions), then

$$\begin{aligned}\lim_{N\to\infty} C^{Ek_1k_2}_{x_1x_2} &= \frac{A}{2}\big[\cos[\omega t + \kappa(x_1 - x_2)] + \cos[\omega t - \kappa(x_1 - x_2)]\big] \\ &= A[\cos(\kappa x_1)\cos(\kappa x_2) + \sin(\kappa x_1)\sin(\kappa x_2)]\cos(\omega t). \qquad (25)\end{aligned}$$

We therefore again observe that eq. (23) holds. In other words, having an ensemble of travelling waves that on average has equal amplitudes is equivalent to having an ensemble of standing-wave modes with equal amplitudes. In this case, having equal amplitude travelling waves corresponds to an isotropic density of sources (i.e. an equal amount of incident wave energy arriving from positive and negative x). Thus, there is an equivalence between equipartition among degenerate modes and an 'isotropic' incidence of waves. One should note that non-degenerate modes need not be equipartitioned for 'isotropy' to exist.

On the other hand, even if the negative and positive arriving waves are not equipartitioned, then

$$\lim_{N\to\infty} C^{Ek_1k_2}_{x_1x_2} = A\cos[\omega t + \kappa(x_1 - x_2)] + B\cos[\omega t - \kappa(x_1 - x_2)]. \qquad (26)$$

If one can pick out only the positive arrival (e.g. by time windowing the signal, as described in Section 3.4) then one can still identify the phase $\omega t - \kappa(x_1 - x_2)$ and hence determine a phase traveltime ω/κ.

4.2 Waves on a periodic square of size L

In this case, $\mathcal{L} = c^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ and the eigenfrequencies can be expressed as $\omega^2_{lm} = \pi^2c^2(l^2 + m^2)/L^2$ for integer l and m. Periodic boundary conditions imply that each ω_{lm} has four modes given by all products of $\{\sin(\kappa_l x), \cos(\kappa_l x)\}$ and $\{\sin(\kappa_m y), \cos(\kappa_m y)\}$. Furthermore, there is additional degeneracy due to the fact that $l^2 + m^2$ can yield the same number for different pairs of (l, m), for example, (7, 1), (5, 5), (1, 7) yield identical $\omega_{lm} = 5\sqrt{2}\pi c/L$. For this example, then, there are degenerate 12 modes.

Expressing these modes as travelling waves gives, for example,

$$\begin{aligned}4\cos(\kappa_l x_1)\cos(\kappa_m x_2)\cos(\omega_{lm} t) &= \cos[\omega_{lm} t - (\kappa_l x_1 + \kappa_m x_2)] \\ &\quad + \cos[\omega_{lm} t + (\kappa_l x_1 + \kappa_m x_2)] \\ &\quad + \cos[\omega_{lm} t + (\kappa_l x_1 - \kappa_m x_2)] \\ &\quad + \cos[\omega_{lm} t - (\kappa_l x_1 - \kappa_m x_2)]. \qquad (27)\end{aligned}$$

As in Section 4.1, each of the degenerate travelling waves having equal amplitudes (on average) is equivalent to equipartition among the degenerate modes. These travelling waves having equal amplitudes means an equal amount of wave energy arriving from the specified directions. One could describe this as 'discretely isotropic', where 'isotropic' refers only to those discrete directions in which waves are allowed to travel. For the $\omega_{lm} = 5\sqrt{2}\pi c/L$ example given earlier, there are 12 of these directions and these directions are not evenly spaced in azimuth.

In the limit of short wavelength modes, $l^2 + m^2 \to \infty$, the number of degenerate travelling waves approaches ∞ and the directions become approximately evenly spaced. This fact can be understood with the following argument. Since $l^2 + m^2 = r^2$ is the equation of a circle in the (l, m) plane, solutions for modes can be described by the intersection of a circle of radius $r = \omega_{lm}L/(\pi c)$ with integer gridpoints. For example, $r = 5\sqrt{2}$ yields the 12 solutions (± 7, ± 1), (± 5, ± 5), (± 1, ± 7) discussed earlier, and the direction of incidence of those 12 waves are the 12 vectors given by the respective coordinates (see Fig. 1a). When $r \to \infty$, the circle has the

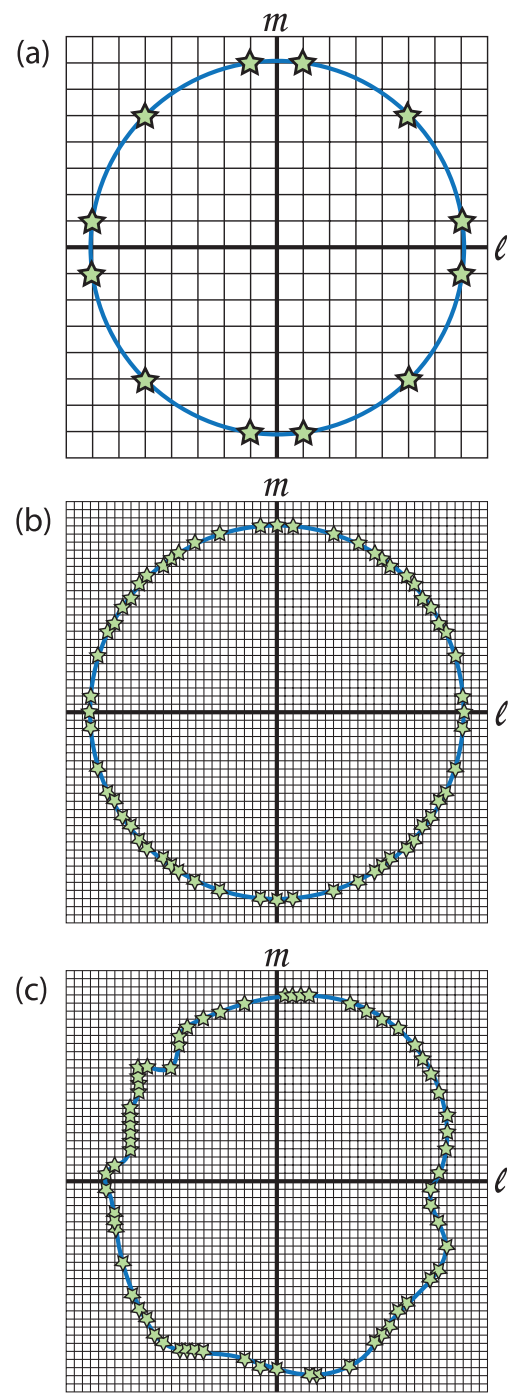

Figure 1. Graphical depiction of solutions for degenerate (travelling wave) modes on a square. Each gridpoint corresponds to a mode. Intersections of the curve $r = \omega_{lm}L/(\pi c)$ with gridpoints denote modes with the same degeneracy (denoted by stars). (a) When $r = 5\sqrt{2}$, the circle intersects 12 gridpoints. (b) When r is large, the modes are approximately evenly spaced. (c) When c is not constant, modes are no longer guaranteed to be evenly spaced. Intersections for cases (b) and (c) are approximate.

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opportunity to intersect many more gridpoints, and these gridpoints will become (approximately) evenly spaced in azimuth, especially if approximate intersection is allowed (i.e. approximately degenerate modes are counted, see Fig. 1b). Thus, we observe that in the limit of short wavelengths, equipartition of modes is equivalent to isotropic incidence of travelling waves.

One should note that if velocity structure is non-uniform [e.g. if $c = c(\theta)$ has a dependence on azimuth, θ], then $r(\theta) = \omega_{lm}L/[\pi c(\theta)]$ is no longer constant. For short wavelength modes, the intersection of this curve with gridpoints is no longer expected to be approximately evenly spaced in azimuth (see Fig. 1c). Equipartition among degenerate modes will therefore no longer imply isotropic incidence of waves. Instead, a very specific non-isotropic incidence of waves will imply equipartition, and only this special case will result in satisfying eq. (23).

4.3 Waves on the Earth

Elastic waves on the Earth (including high-frequency body waves) can be described in terms of (spheroidal and toroidal) normal modes (e.g. Dahlen & Tromp 1998) so the formulation of Section 3 can still be used. As with the previous two examples, modes are discrete, with the longest period modes having a small number of (approximately) degenerate members (sometimes called multiplets). For these longest period modes (e.g. 400–4000 s), as in the examples of Section 4.2, equipartition of these degenerate modes would imply an 'isotropic' distribution of waves only in the discrete directions allowed by the Earth.

On the other hand, for short period modes (e.g. 10 s) whose wavelengths (e.g. 40 km) are very small compared to the Earth's radius, the Earth can be approximated as flat and hence the situation is like that for the short wavelength modes on a periodic square of Section 4.2, except that there are multiple mode branches instead of just one. Thus, a large number of modes are degenerate and equipartition implies 'nearly' isotropic incidence of waves. As in Section 4.2, the qualifier 'nearly' is needed because the Earth has a (mostly weakly) heterogeneous velocity structure rather than being perfectly homogeneous. It should be noted that 'nearly' isotropic incidence does not necessarily imply equipartition if there are multiple mode branches that are (approximately) degenerate. For example, degenerate higher order modes could have very low amplitudes and not be equipartitioned while degenerate fundamental modes could be equipartitioned, resulting in 'nearly' isotropic incidence but only partial equipartition. Time windowing as in Section 3.4 could be applied to consider only the fundamental branch.

Traditional ambient noise tomography applications on the Earth (e.g. Shapiro & Campillo 2004; Sabra *et al.* 2005; Shapiro *et al.* 2005) typically focus on relatively short-period waves (e.g. 5–40 s) excited by microseism energy. The implication of Section 3 is that if these short-period travelling waves are (nearly) isotropic, then these degenerate modes are equipartitioned and the cross correlation and Green's function will be related by eq. (23). If degenerate-mode waves are only partially equipartitioned (e.g. there exist higher order modes approximately degenerate with the fundamental branch of interest, as would be expected), a partial Green's function may still be obtained from the cross correlation as described in Section 3.4 by use of time windowing, but quantifying the degree of success is difficult within the framework presented here. If waves are not equipartitioned, none of the analysis discussed here will have bearing on how cross correlation measurements can be related to the Green's function. Analysis like that of Cox (1973), Harmon *et al.* (2008) or Tsai (2009), which describe a framework for examining non-isotropic incidence of waves with potentially non-uniform velocity structure, is better suited for quantifying the degree to which cross correlation measurements can still yield useful information related to the Green's function. Since it is known that microseism energy is primarily generated in the oceans (which are not isotropically distributed), the success of recent noise tomography applications (e.g. Shapiro & Campillo 2004; Shapiro *et al.* 2005) is therefore still best understood through those alternative means. Nonetheless, the above analysis clarifies a number of points regarding noise cross correlation on the Earth, including the fact that equipartition need not be satisfied among all modes for a useful relationship between the cross correlation and Green's function to exist.

5 CONCLUSIONS

In this work, we take a modal approach and demonstrate relationships between the cross correlation and the Green's function, specifically examining the differences between the four cases where modes are equipartitioned or not and degenerate or not. We find that in certain cases waves need not be randomly excited and equipartition does not need to be strictly satisfied for a useful relationship to exist between the cross correlation and the Green's function. Specifically, if no modes are degenerate then equipartition implies the standard relationship holds even with a completely deterministic system without random excitations; if there are degenerate modes, equipartition must be accompanied by random phases for the standard relationship to hold; if modes are not equipartitioned but there is also no degeneracy, the relationship holds in a restricted sense to each frequency; finally, if modes are degenerate and not equipartitioned, partial Green's function phase information can sometimes be retrieved depending on whether partial equipartition exists. Applying this analysis to a few simple examples explains some aspects of noise cross correlation on the Earth, although quantitative analysis of the degree of success of noise tomography applications remains better understood through other means.

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