Cross-Pol: simple estimates

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1 Summary

- Getting XP to be a factor of 10 below the cosmological signal requires rotations of less than 0.3^o. The requirement is less stringent for the GW signal, 1.3^o is enough for $T/S = 0.1$.
- Differential XP across detectors can be constrained by demanding that all detectors see the same. Two detectors observing a common area for a week can be calibrated to approximately 2^o.
- The simultaneous calibration of many detectors reduces the above by $\sqrt{2}$.
- If one has a model for the rotation across the focal plane with N_p parameters, then the error scales down by $\sqrt{N_p/N_d}$ where N_d is the number of detectors.
- To the extent that the residual for each detector is random, there will be an additional reduction because of averaging by maybe as much as another $\sqrt{N_d}$.
- The overall rotation common to all detectors can be constrained because it induces a cross correlation between B and both T and E. Most of the signal comes from $E-B$. The overall rotation can be constrained to 0.04^o. This effect appears to be easier to control. A possible complication is E-B separation.
- The overall rotation is easier to determine because one is combining the information of N_d detectors so one gets an improvement of $\sqrt{N_d}$ in the noise and the angle constraint.
- Even if one does not have a model for the XP across the focal plane, the process of mapmaking may provide a simplification where not all the XP angles for each detector are needed but only a smaller number of combinations. This may improve the situation substantially.
- Perhaps an obvious point: one should measure $E-B$ and $T-B$ cross spectra as a monitor of systematics .

2 Leakeage from Cross-Pol

To make a simple order of magnitude estimate of the effect we can reproduce Brad's simulations. Assume the flat sky approximation and take that the polarization reference frame was rotated by an angle α so that,

$$\tilde{Q} + i\tilde{U} = e^{2i\alpha}(Q + iU) \tag{1}$$

with $\tilde{Q}$ and $\tilde{U}$ represent the measured values and Q and U are the real ones.

In the flat sky approximation it is easy to relate the corresponding measured $\tilde{E}$ and $\tilde{B}$ to the underlying ones,

$$\begin{aligned}\tilde{E}(\mathbf{l}) &= \cos 2\alpha E(\mathbf{l}) + \sin 2\alpha B(\mathbf{l}) \\ \tilde{B}(\mathbf{l}) &= -\sin 2\alpha E(\mathbf{l}) + \cos 2\alpha B(\mathbf{l}).\end{aligned} \tag{2}$$

As a result of this mixing the power spectra of E and B are modified. Also very importantly there is now an additional cross correlation between E and B and between B and T that were not present before. The modified power spectra are given by:

$$\begin{aligned}\tilde{C}_l^B &= \cos^2(2\alpha)C_l^B + \sin^2(2\alpha)C_l^E \\ \tilde{C}_l^E &= \sin^2(2\alpha)C_l^B + \cos^2(2\alpha)C_l^E \\ \tilde{C}_l^{EB} &= \frac{1}{2}\sin(4\alpha)(C_l^B - C_l^E) \\ \tilde{C}_l^{TB} &= -\sin(2\alpha)C_l^{TE} \\ \tilde{C}_l^{TE} &= \cos(2\alpha)C_l^{TE}\end{aligned} \tag{3}$$

From the above equation it is easy to determine the level of cross-pol that can be tolerated. Figure 1 shows the comparison of the cosmological signal (GW with $T/S = 0.1$ + lensing) with the B modes generated by the leakege from E. For $l = 100$ the contamination produced by a rotation of 1.3^o a factor of ten below the GW signal (for T/S=0.1). The lensing signal is tougher to get, requiring an $\alpha < 0.35^o$ to be a factor of 10 below the signal. In the C_ls level of contamination scales as α^2 and the amplitude of the GW scale as T/S.

We will use this as a benchmark for the typical rotations that are allowed. In what follows we will divide the effect as a relative rotation between different detectors and an overall average offset. We will try to calibrate each of them independently.

3 Relative offset

The relative offset can be calibrated by demanding that different detectors see the same polarization signal. First we can do an order of magnitude estimate assuming two detectors observe the same patch of sky for a time t_o. Assume:

$$(\tilde{Q} + i\tilde{U})_k = e^{2i\alpha_k}(Q + iU) + N_k, \tag{4}$$

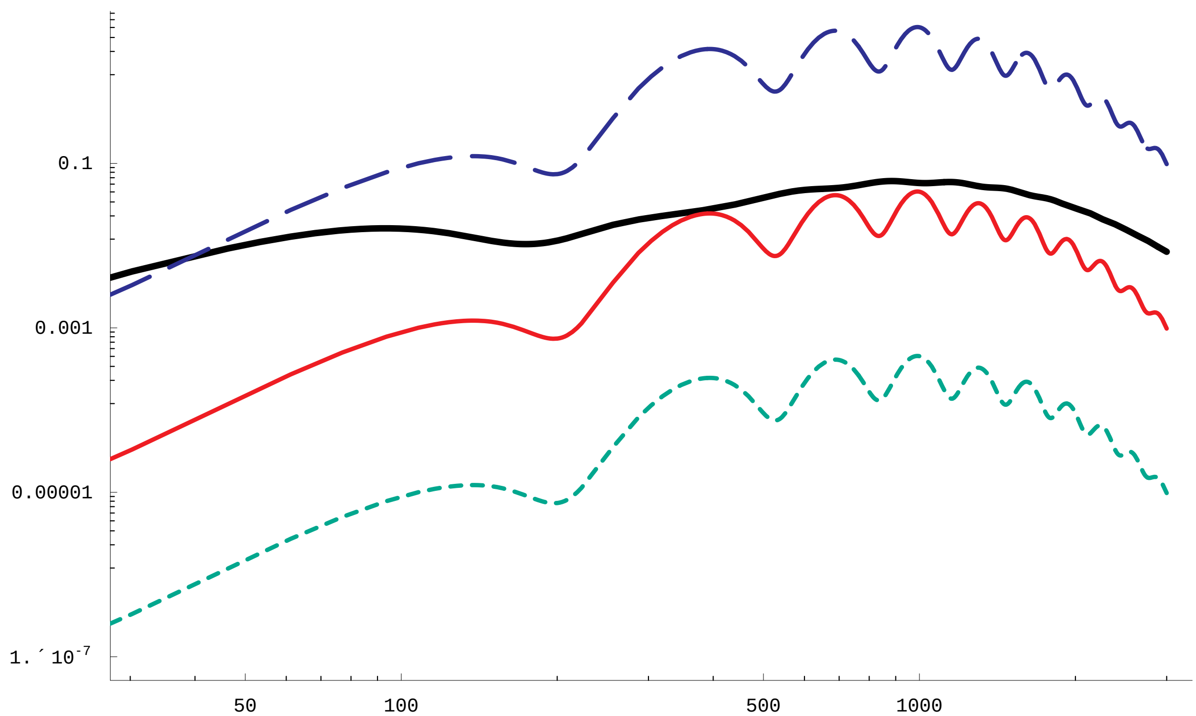

Figure 1: Power spectrum of B modes generated by XP compared to cosmological signal. The y axis is $l^2 C_l/2\pi$ in μK^2. Black curve is for cosmological B modes (lensing + GW with T/S=0.1). Three curves for contamination are shown, from top to bottom $\alpha = 10^o,\ 1^o$ and 0.1^o. The contamination curves scale as α^2 and the level of GW signal as T/S.

where $k = 1,2$ runs over the number of detectors and N_k is the noise. The results are identical if one assumes that the two detectors are always looking at the same pixel or that the look at many of them. If there are many pixels Q, U and N are vectors of dimensions N_{pix}.

The easy way to estimate how well you can measure the difference of αs is to consider the difference $d = (\tilde{Q} + i\tilde{U})_1 - (\tilde{Q} + i\tilde{U})_2$. The difference d is a Gaussian variable and one can estimate the expected value of the likelihood, or χ^2 as a function of $\delta\alpha$. The result is:

$$\langle \Delta\chi^2 \rangle = 2\frac{\langle Q^2 + U^2 \rangle}{\sigma^2}\delta\alpha^2 \tag{5}$$

For LCDM $\langle Q^2+U^2 \rangle = 20\ \mu K^2$, and $\sigma^2 = s^2/t_o$ where s is approximately $150\mu K\sqrt{\mathrm{sec}}$. Thus we get an error on α,

$$\sigma_\alpha = 1.8^o \times \frac{s}{150\mu K\sqrt{\mathrm{sec}}} \times (\frac{1\ \mathrm{week}}{t_o})^{1/2} \times \frac{4.4\mu K}{\sqrt{Q^2 + U^2}} \tag{6}$$

So the situation is not that good. One can do a few more extension of the calculation. For example consider the case when you have many detectors, wether there is any improvement. One can calculate again the new expected χ^2. Again the average α cannot be constrained. The answer in this case is (for large numbers of detectors),

$$\langle \Delta\chi^2 \rangle = 4\frac{\langle Q^2 + U^2 \rangle}{\sigma^2}\sum_k \delta\alpha_k^2, \tag{7}$$

where $\delta\alpha_k = \alpha_k - \bar{\alpha}$ and $\bar{\alpha}$ is the average offset which cannot be constrained. From the above formula we see that if all the $\delta\alpha_k$ are parameters they each can be constrained only a factor of $\sqrt{2}$ better than when we compare just two detectors. The reason of the factor of two is that you can think that you are comparing each detector not to another detector but to the combination of all of them. So in the calculation for the simple difference is the same as assuming one of the two detectors had very little noise. So you gain a factor of 2 in $\delta\chi^2$, a factor of $\sqrt{2}$ in σ_α. Another technical note is that I derived the above equation both when everybody is looking at a single pixel and when each detector is covering a map.

Another cross check, the last result should be equivalent to trying to calibrate the cross-pol on one detector using a CMB pixel of known polarization. Assume that we orient our reference frame so that the polarization in that pixel is in $Q = P$. The rotation angle is given by $\tan(2\alpha) = U/Q$. Thus if for the detector the noise in U is σ then

$$\delta\alpha = \frac{1}{2}\sigma/P, \tag{8}$$

which is exactly the above equation in the case of many detectors. If $P \approx 4.4\ \mu K$ and $\sigma \approx 0.2\ \mu K$, this gives $\delta\alpha \approx 1.3^o$.

One gains a bit if one has a model for the deflection that depends on only a few parameters. For example if $\delta\alpha(\mathbf{x}) = a_m x^2 \cos(m\phi)$ where x is say the distance to the center of the focal plane and ϕ the angle around it. One is trying to constrain a_m.

For such a model, by inserting it into the from of $\delta\chi^2$ and seeing what the constrain in a_m would be, one can see that the worse deflection (at the edge of the field) is down from the number when all detectors are being fitted by a factor of $\sqrt{\frac{6}{N_d}}$. In general one would get an improvement proportional to $\sqrt{N_{par}/N_d}$ where N_{par} is the number of parameters in the model and N_d the number of detectors.

Finally to the extent that the residual from each detector is is random, then in the combined map the error will be further reduced by as much as an additional $\sqrt{N_d}$ if each pixel was measured equal amount of times by each detector.

4 Overall Offset

The key to determining the overall offset is that it creates an E-B cross correlation and a T- B cross correlation. Of course being able to detect this relies on being able to separate E and B in a finite patch. We will ignore this issue.

Given the formulas of the cross correlations $E-B$ and $T-B$ in equation (3) we can ask the question for what α do they become detectable. The signal to noise is each of them is:

$$\begin{aligned}(S/N)^2|_{E-B} &= 4\alpha^2 f_{sky}\sum_l (2l+1)\frac{(C_l^E - C_l^B)^2}{(C_l^B + w_P^{-1})(C_l^E + w_P^{-1})} \\ (S/N)^2|_{T-B} &= 4\alpha^2 f_{sky}\sum_l (2l+1)\frac{(C_l^{TE})^2}{(C_l^B + w_P^{-1})(C_l^T + w_T^{-1})}\end{aligned} \tag{9}$$

Plugging in the numbers for the EBEX patch I get,

$$\begin{aligned}(S/N)^2|_{E-B} &= (\frac{\alpha}{0.04^o})^2 \\ (S/N)^2|_{T-B} &= (\frac{\alpha}{0.11^o})^2\end{aligned} \tag{10}$$

So all the information comes from $E-B$ and that is because the cross correlation coefficient between T and E is not one to start with. It appears that α can be measured quite accurately this way.

One can do a more sophisticated calculation. One can calculate the Fisher matrix for the power spectra and α as an additional parameter. It is easy to prove that the Fisher matrix is diagonal between the power spectra and α and obtain the expected error bar on α when the bandpowers are fitted at the same time. I have computed all the elements of the Fisher matrix and obtained the expected constraints on α. By far most of the information comes from $E-B$, as the full result differs from what one gets by measuring only $E-B$ by less than 1% .

5 Map Making

Even if one does not have a model for the XP across the focal plane the map making process may provide a simplification. For example if all detectors are weighted equally across the map, then only the average α will be relevant. This of course will not be the case but it may still be true that the number of independent numbers needed to characterize the XP in the final map is less that one per detector. It should be possible to estimate this using the know scan at least for a simple co-add map making procedure. This might be something that Nicola can do with his simulations.