1 Pointing requirements at the level of the final map

Assume there is a pointing error given by $\delta\theta$ for each pixel in the map. **This is the final error at the level of the map. The error on an individual scan can be larger provided there is sufficient "averaging-down"**. Given an assumption of the statistical properties of $\delta\theta$ one can calculate the B modes created from the E modes. This is completely analogous to lensing.

For simplicity we will assume that the power spectrum of $\delta\theta$ is white up to some coherence scale and that the x and y components of the deflection angle are independent,

$$\langle \delta\theta_i(\mathbf{l}_1)\delta\theta_i^*(\mathbf{l}_1)\rangle = (2\pi)^2\delta_{ij}\delta^D(\mathbf{l}_1 - \mathbf{l}_2)C_l^\theta = (2\pi)^2\delta_{ij}\delta^D(\mathbf{l}_1 - \mathbf{l}_2)\frac{2\pi\sigma_\theta^2}{l_s^2}e^{-l^2/2l_s^2}, \tag{1}$$

where i, j stand for the different components of the deflection angle and l_s encodes the coherence length. The power spectrum is normalized so that

$$\sigma_\theta^2 = \int \frac{d^2l}{(2\pi)^2}C_l^\theta \tag{2}$$

Important physical point: the effect will be maximum for a coherence length close to the beam scale. To see this first consider what happens when you increase the coherence length. It is important to note that a uniform translation of all pixels will not convert E to B. As a result if we consider a pointing error field that has a large coherence length, E modes of wavelength smaller than this will not contribute to the conversion. If less ls are being converted then as we increase the coherence length the effect is reduced. This is especially true because the power spectrum of the derivative of E is very blue, so as you take out the contribution from the high ls you loose quite a bit.

The l^2C_l of the B contamination increase like l^2 up to the coherence length of $\delta\theta$ (l_s), roughly scaling as

$$\frac{l^2\Delta C_l^B}{2\pi} \approx \frac{1}{2}\sigma_\theta^2(l/l_s)^2\int^{l_s}\frac{dl}{l}\frac{l^4C_l^E}{(2\pi)}. \tag{3}$$

A comparison between ΔC_l^B for various values of l_s and the B modes can be seen in the first figure of the mathematica output. As a result the lensing signal is the much more affected. Note that here is a simple interpretation for equation (3). The integral just gives the expectation value of the gradients of the stokes parameters. The pointing error induces an error in the map of the form $\delta Q = \delta\theta \times \nabla Q$ and the same for U. This term creates additional power in the map, half of which goes to E and half to B, the reason for the 1/2 in equation (3). This happens on small scales so the power spectrum has a white noise form on large scales. The additional B

power has to integrate to $1/2\sigma_\theta^2\langle|\nabla Q|^2 + |\nabla U|^2\rangle$ but with only modes with $l < l_s$ contributing to the gradient, exactly what (3) says.

The second and third figures show the ratio between ΔC_l^B and the cosmological C_l^B(assuming T/S =0.1) at $l = 100$ and $l = 1000$ as a function of l_s when $\sigma_\theta =$ 1 arcmin. The worst ratio comes for $l_s \sim 500$. For that choice one has

$$\frac{\Delta C_l^B}{C_l^B}|_{l=100,l_s=500} \approx \frac{0.25\ \sigma_\theta^2}{(1\ \text{arcmin})^2} \times (\frac{T/S}{0.1})^{-1}, \tag{4}$$

and

$$\frac{\Delta C_l^B}{C_l^B}|_{l=1000,l_s=500} \approx \frac{4\ \sigma_\theta^2}{(1\ \text{arcmin})^2}. \tag{5}$$

Thus to get the contamination to be 1/10th of the signal for the lensing one needs $\sigma_\theta \sim 0.16$ arcmin ~ 9 arcsec. For the GW with T/S of 0.1 the requirement is $\sigma_\theta \sim 0.6$ arcmin ~ 37 arcsec. For the GW with T/S of 0.01 the requirement is $\sigma_\theta \sim 0.2$ arcmin ~ 10 arcsec.

1.1 Some Details of the calculation

We expand both Q and U in series so that the observed Q ($\tilde{Q}$) and U ($\tilde{U}$) are given by,

$$\begin{aligned} \tilde{Q} &= Q + \delta\theta_i\partial_i Q + \frac{1}{2}\delta\theta_i\delta\theta_j\partial_{ij}^2 Q + \cdots \\ \tilde{U} &= U + \delta\theta_i\partial_i U + \frac{1}{2}\delta\theta_i\delta\theta_j\partial_{ij}^2 U + \cdots \end{aligned} \tag{6}$$

We use the above expressions to go to Fourier space and compute $\tilde{Q}(\mathbf{l})$ and $\tilde{U}(\mathbf{l})$ and combine them to form $\tilde{B}(\mathbf{l}) = -\sin 2\phi_l \tilde{Q}(\mathbf{l}) + \cos 2\phi_l \tilde{U}(\mathbf{l})$. Squaring $\tilde{B}(\mathbf{l})$ we then calculate the change in B power spectrum.

$$\Delta C_l^B = \int \frac{d^2l'}{(2\pi)^2} C^\theta_{\mathbf{l}-\mathbf{l}'}\ \sin^2 2(\phi_l - \phi_{l'})\ l'^2 C_{l'}^E \tag{7}$$

A simple check of what happens when the coherence length increases, that is when $l_s \to 0$. In that limit $C^\theta_{\mathbf{l}-\mathbf{l}'}$ approaches $\delta^D(\mathbf{l}-\mathbf{l}')$ and so the integral tends to zero because $\sin^2 2(\phi_l - \phi_{l'}) \to 0$. This is simply because a constant shift cannot convert E to B and for modes much smaller than the coherence length of the deflection the deflection is effectively constant.

In the other limit, when $l_s \to \infty$ C_l^θ is effectively constant so

$$\frac{l^2 \Delta C_l^B}{2\pi} \approx \frac{1}{2}\sigma_\theta^2 (l/l_s)^2 \int^{l_s} \frac{dl}{l} \frac{l^4 C_l^E}{(2\pi)}, \tag{8}$$

with the 1/2 coming from the average of $\sin^2 2(\phi_l - \phi_{l'})$.

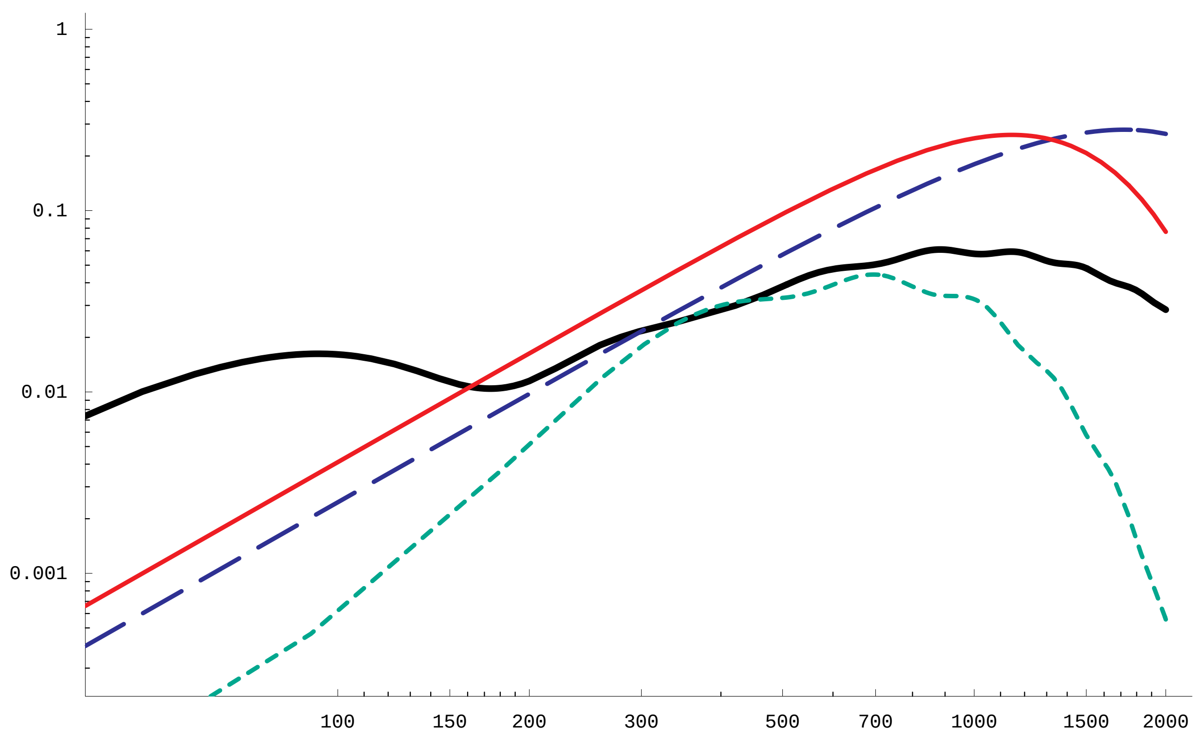

Figure 1: Power spectrum of B modes generated by pointing errors compared to cosmological signal. The y axis is $l^2 C_l/2\pi$ in μK^2. Black curve is for cosmological B modes (lensing + GW with T/S=0.1). Three curves for contamination are shown, from top to bottom l_s=500,1000 and 100, l_s=500 is close to the worse case scenario. Deflection angle RMS was taken to be σ_θ of 1 arcmin. The contamination curves scale as σ_θ^2

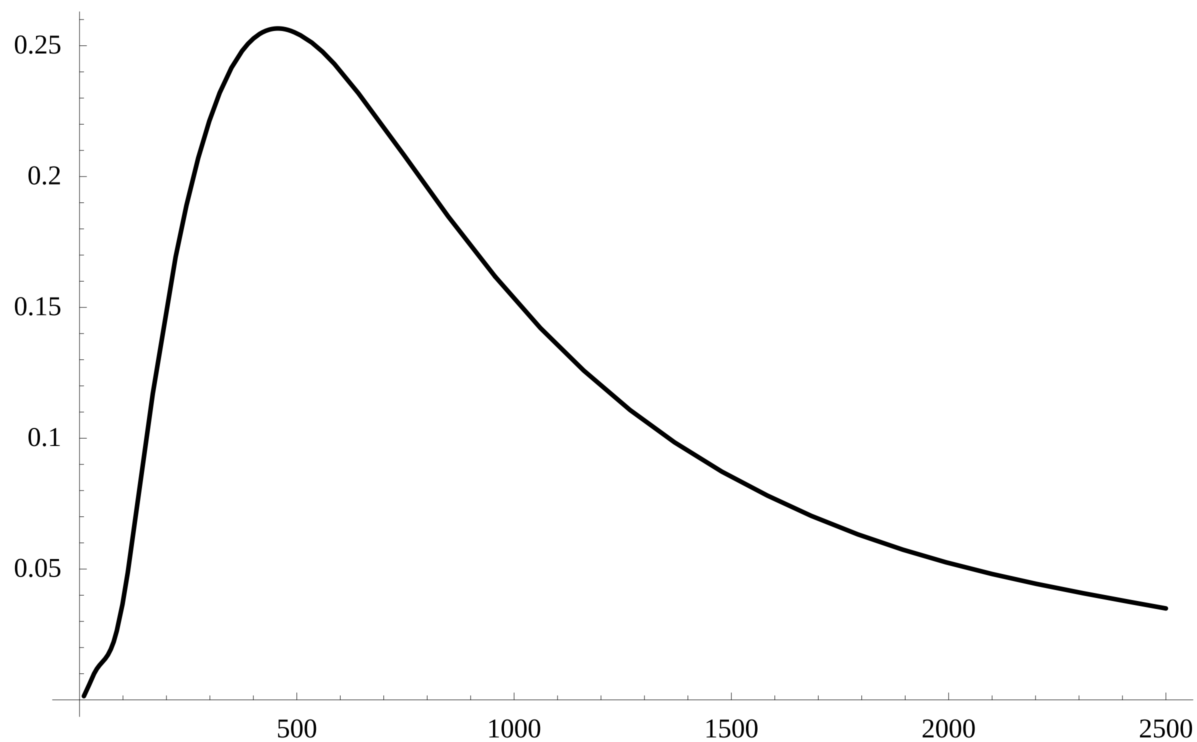

Figure 2: Ratio Cls of B modes from contaminant to cosmological signal as a function of l_s for l=100 (relevant for the GW signal). Assumed σ_θ=1 arcmin. Ratio of scales as σ_θ^2

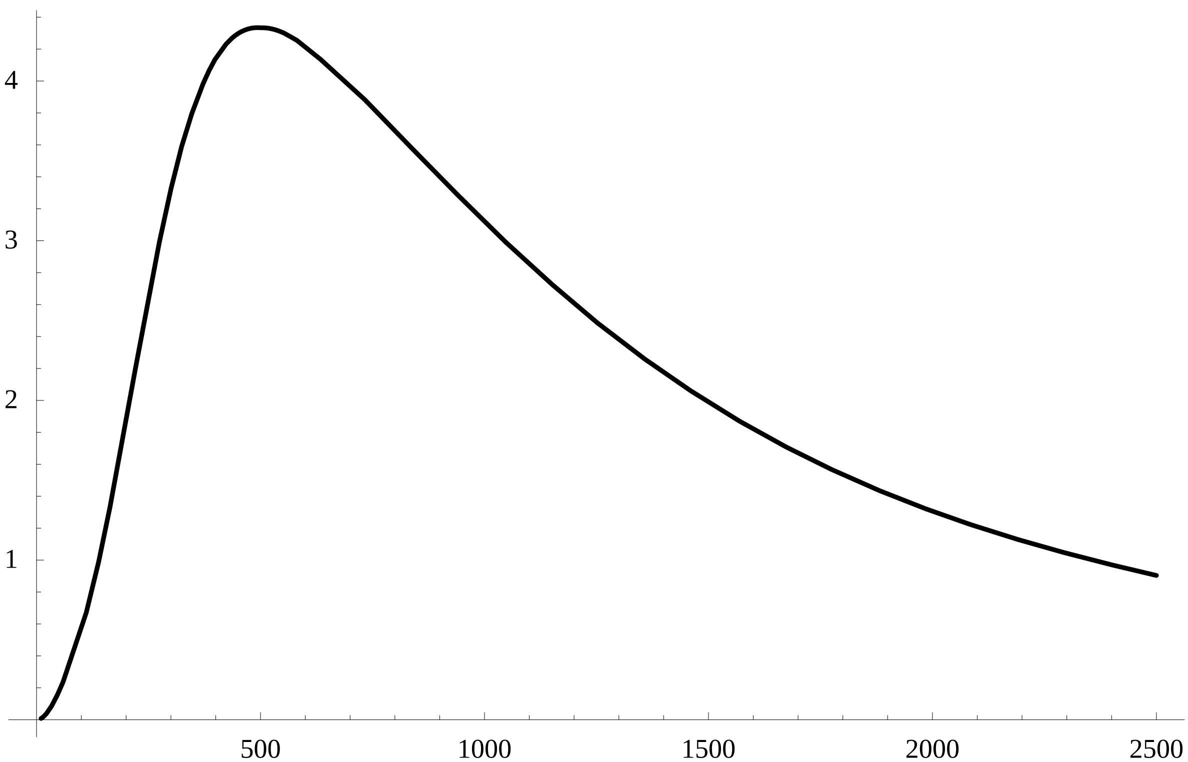

Figure 3: Ratio Cls of B modes from contaminant to cosmological signal as a function of l_s for l=1000 (relevant for the lensing signal). Assumed σ_θ=1 arcmin. Ratio of scales as σ_θ^2

2 From scan to map

We could think of the final error in the map as follows. At any given time during the scan one thinks that one is pointing in a particular direction but the pointing reconstruction has an error $\delta\theta_{point}$. Furthermore one is using some interpolation scheme when assigning the measured polarization to a particular pixel. For example one could be assigning the measurement just to the nearest pixel. This results in an error $\delta\theta_{pix}$. Because of these two errors one is assigning a Q equal to $Q + (\delta\theta_{point} + \delta\theta_{pix}) \cdot \nabla Q + \cdots$ to a given pixel rather than Q. same is valid for U

As a result the Stokes parameters at a pixel in the final map are something like:

$$\hat{Q} = Q + 1/N \sum_i (\delta\theta_{point} + \delta\theta_{pix})_i \cdot \nabla Q + \cdots, \tag{9}$$

where the sum is over the N times the pixel was measured (assuming for simplicity that they all went to the map with equal weights). The same formula applies for U.

We will consider both contributions separately in the next sections.

2.1 Pointing noise in each scan

Assume the map is produced by scans of length L over a time t_{sc}. At the edge of each scan one gets the pointing with a star camera and extrapolates using gyros to points inside the scan. The pointing error in each component across the scan is given by:

$$\delta\theta = \delta x_0 + \frac{t}{t_{sc}}(\delta x_L - \delta x_0) + \sum_n a_n \sin k_n t, \quad k_n = \frac{n\pi}{t_{sc}} \tag{10}$$

where δx_0, δx_L and a_n are random variables and t is time. We are calculating the errors in the position of the center of the focal plane along the scan. Detectors are then referenced to that.

Under the assumption that the error in the gyros is a random walk the a_ns are independent Gaussian random variables. The statistical properties of δx_0 and δx_L are mainly determined by the properties of the star camera system. In particular:

$$\langle a_n a_m \rangle = \delta_{n,m}\sigma_n^2 = \delta_{n,m}\frac{w_g t_{scan}}{(n\pi)^2}, \tag{11}$$

with w_g a property of the gyros, the square of the random walk dispersion for a given time interval, and t_{scan} is the time taken by the scan to cross the distance L.

If the pointing error of the star camera is given by σ_{sc} then the inverse of the δx_0-δx_L covariance matrix is:

$$C^{-1} = \begin{pmatrix} \sigma_{sc}^{-2} + (w_g t_{scan})^{-1} & -(w_g t_{scan})^{-1} \\ -(w_g t_{scan})^{-1} & \sigma_{sc}^{-2} + (w_g t_{scan})^{-1} \end{pmatrix}. \tag{12}$$

The above formulas can be used to generate pointing error time streams that can be incorporated to full scale simulatons. They are valid for both x and y components of the deflection angle. A derivation of the above formulas can be found in section 2.5.

The variance of the error for one pixel and one scan is:

$$\langle\delta\theta^2\rangle = \langle(\delta x_0)^2\rangle(\frac{t}{t_{sc}})^2 + \langle(\delta x_L)^2\rangle(1-\frac{t}{t_{sc}})^2 + 2\langle\delta x_0 \delta x_L\rangle(\frac{t}{t_{sc}})(1-\frac{t}{t_{sc}}) + \sum_n \sigma_n^2 \sin^2 k_n t. \tag{13}$$

We show a plot of this function in figure (4). A similar formula can be obtained for the correlation of $\delta\theta$ at two points during the scan. Note that the two errors are highly correlated. This is so because the star camera noise affects all points and due to the random walk nature of the gyros, its errors are dominated by the lowest n modes ($\sigma_n^2 \propto n^{-2}$).

2.2 Averaging down the pointing noise

Although the errors in each scan are highly correlated, we will take the noise in different scans to be uncorrelated. As a result there could be significant averaging down of the error as each pixel is scanned multiple times. We will estimate this averaging in this section. **The results depend on details of the focal plane and scan. Here we will make some crude assumptions to be able to put some numbers. We can probably put the full formulas into Nicola's pipeline if the following arguments need to be made more precise.**

As figure 4 shows the rms pointing error varies across the scan. If a pixel is hit by multiple scans it will probably be hit at different points along the scan. To simplify matters we will just replace the x dependent rms by the corresponding average. This will simplify expressions. Note however that I will not assume that the errors along the scan are all independent. In essence I will work in the opposite limit, just assume that all pixels in a given scan have the same pointing error.

The average variance across the scan is:

$$(\sigma_\theta^2)_1 = \frac{1}{6}\left(4\sigma_{sc}^2 + w_g t_{scan} - \frac{2\sigma_{sc}^4}{2\sigma_{sc}^2 + w_g t_{scan}}\right). \tag{14}$$

For our current gyros this gives $(\sigma_\theta^2)_1 = (9.5 \text{ arcsec})^2$ and for the better ones $(\sigma_\theta^2)_1 = (4.3 \text{ arcsec})^2$. If we now assume that a pixel is observed many times during independent scans, and that in each scan it is measure by δN_d detectors, the final variance at that pixel will be:

$$\sigma_\theta^2 = \frac{\sum \delta N_d^2 (\sigma_\theta^2)_1}{(\sum \delta N_d)^2} \equiv \frac{(\sigma_\theta^2)_1}{\sqrt{N_{eff}}} \tag{15}$$

Now we need to estimate how N_{eff}.

Consider a given scan. As the focal plane moves, each detector will swipe a line of length L and width dx_{pix}, where dx_{pix} is the linear size of each pixel. If the

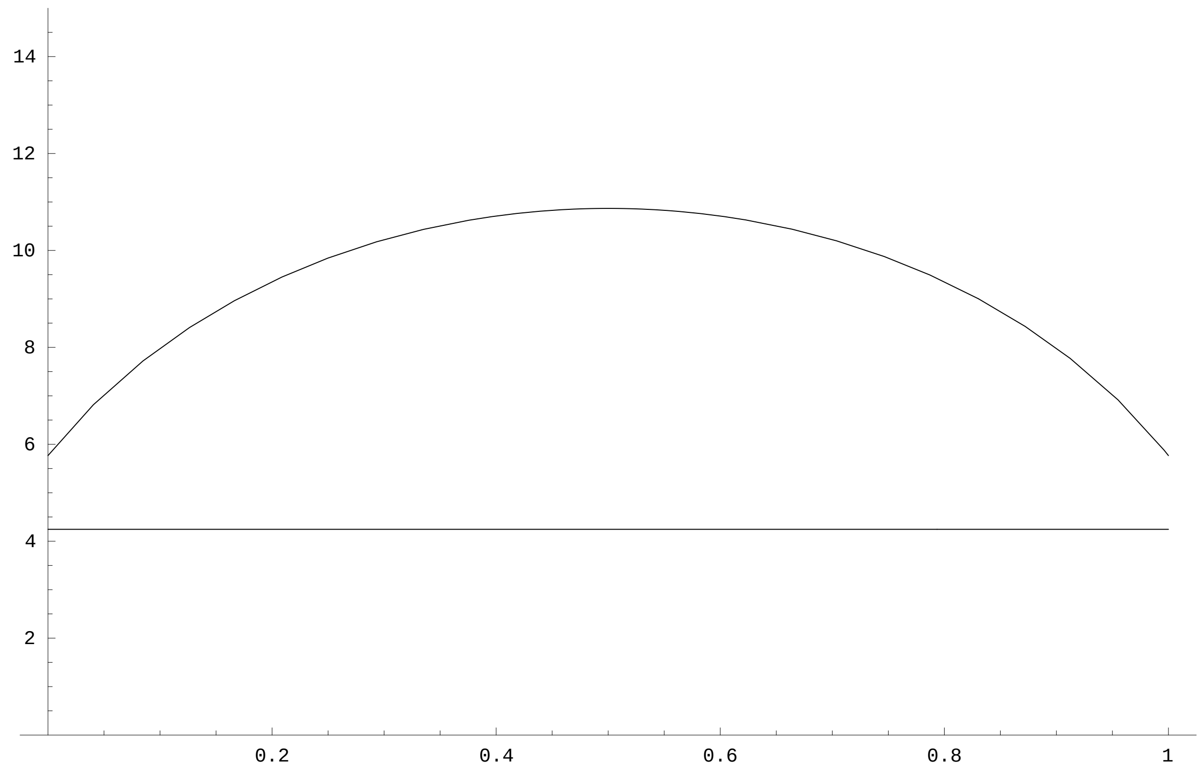

Figure 4: RMS pointing error in arcseconds as a function of position across the scan. The star camera error assumed is $\sigma_{sc} = 6$ arcsec. the top curve is for $w_g = 0.0667$ deg/$\sqrt{\mathrm{hr}}$ and the bottom one for $w_g = 0.001$ deg/$\sqrt{\mathrm{hr}}$.

sampling rate is fast enough then all of the solid angle $L \times dx_{pix}$ will be covered by that detector. If the sampling rate is low, only a fraction $f = dx_{sample}/dx_{pix}$ of the pixels in the row will get hit, where dx_{sample} is the separation between samples.

On a given scan all the different detectors in the focal plane will share the same pointing error. The combination of all detectors will swipe a larger solid angle as the scan proceeds that a single detector. How much larger depends on the location of detectors across the focal plane, as the paths of multiple detectors in the same row will overlap. For simplicity I will just consider square array of detectors, $N_1 \times N_1 = N_{total}$. Thus in one scan the total solid area covered by all the detectors $d\Omega$ is:

$$d\Omega = N_1 \times L \times dx_{pix} \times f. \tag{16}$$

After the full flight the total solid angle covered will be:

$$\Omega = N_1 \times L \times dx_{pix} \times f \times \frac{t_{flight}}{t_{scan}}. \tag{17}$$

The above estimate is not quite correct because two neighboring scans share one star camera pointing so they are not fully independent. To the extent that the noise is dominated by the gyros the above approximation is correct. If not one should modify the formulas to take into account the correlations. The change will not be large, should not decrease N_{eff} by more than a factor of 2.

To estimate N_{eff} we will assume that N_{eff} is the same for all pixels in the final map. Of course this is not really true. If we call Ω_{map} the solid angle of the final map then $N_{eff}\Omega_{map} = \Omega$, so

$$\begin{aligned} N_{eff} &= \Omega_{map}^{-1} \times N_1 \times L \times dx_{pix} \times f \times \frac{t_{flight}}{t_{scan}} \\ &\approx 622 \ \frac{350 \text{ deg}^2}{\Omega_{map}} \times \frac{N_1}{30} \times \frac{L}{18^o} \times \frac{dx_{pix}}{1'} \times \frac{f}{1} \times \frac{t_{flight}}{1 \text{ week}} \times \frac{25 \text{ sec}}{t_{scan}} \end{aligned} \tag{18}$$

For our current gyros we get $\sigma_\theta = (\sigma_\theta)_1/\sqrt{N_{eff}} \approx 0.4$ arcsec, extremely small. One can also estimate the average hits per pixel

$$\begin{aligned} N_{hits} &= \Omega_{pix} \times \Omega_{map}^{-1} \times N_{total} \times f_{sample} \times t_{flight} \\ &\approx 73000 \ \frac{350 \text{ deg}^2}{\Omega_{map}} \times \left(\frac{dx_{pix}}{1'}\right)^2 \times \frac{N_{total}}{400} \times \frac{f_{sample}}{380Hz} \times \frac{t_{flight}}{1 \text{ week}} \end{aligned} \tag{19}$$

2.3 Comparison with the scanning code

Using the EBEx scanning code in which the arrangement of the seven decagon wafers has been modelled, one can obtain the hit and the scan crossing counts in order to directly check scaling relations of Eq. (18) and Eq. (19). Figure 5 shows an example of a scan crossing count map for the 150GHz channel (396 detectors) using the parameters $t_{flight} = 0.9$day, $t_{scan} = 25.7$s ($t_{chop} = 51.4$s), $f_{sample} = 300$Hz, two

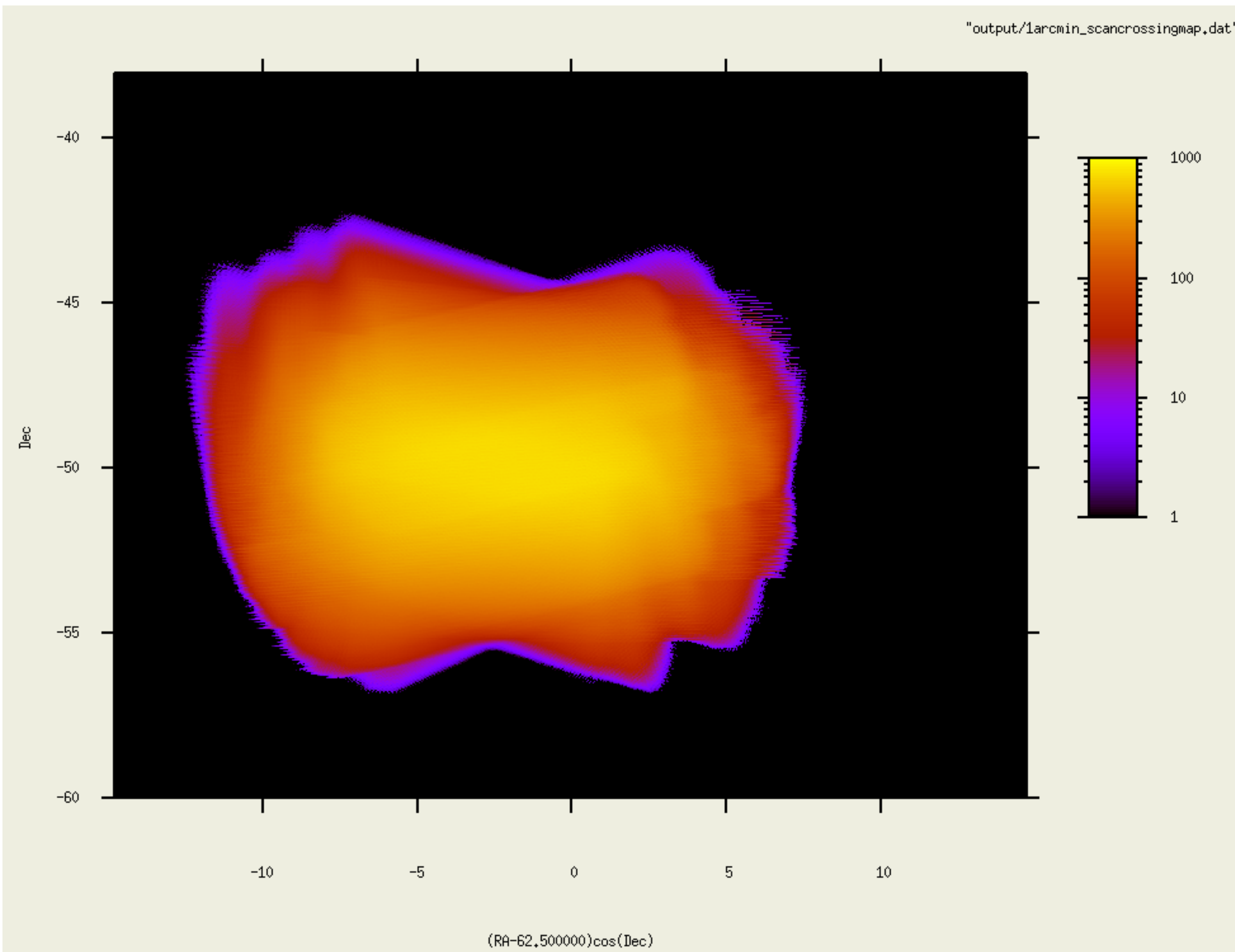

Figure 5: An example of a scan crossing count map for the 150GHz channel (396 detectors) using the parameters $t_{flight} = 0.9$day, $t_{scan} = 25.7$s ($t_{chop} = 51.4$s), $f_{sample} = 300$Hz, two focalplane hits per sample, $dx_{pix} = 1'$ and $L = 17.7^o$. The total scan area is 240deg^2 giving an average scan crossing count of 250 per pixel.

focalplane hits per sample, $dx_{pix} = 1'$ and $L = 17.7^o$. The total scan area is 240deg^2 giving an average scan crossing count of 250 per pixel.

The scaling with pixel size of the hit and scan crossing counts is shown in figure 6 for the three EBEx channels. The scaling relation for the average hit count, Eq. (18), in is perfect agreement with the results from the scanning code, being a simple average of the total number of hits. The scaling relation for the average scan crossing count Eq. (19) is also in very good agreement with the results from the scanning code. In order to partially fix up the square array approximation above, N_1 has been replaced by the number of rows of detectors parallel to the scanning direction (59, 42, and 25 in the case of the 150, 250 and 420GHz detectors respectively).

2.4 Pixelization noise

If one is using a nearest neighbor interpolation the error in a Stokes parameter for a given $\delta\theta_{pix}$ is

$$\delta Q = \delta\theta_{pix} \cdot \nabla Q. \tag{20}$$

If we assume that $\delta\theta_{pix}$ is uniformly distributed inside the pixel and that the linear size of a pixel is dx_{pix} then

$$\sigma_\theta^2 = \frac{1}{dx_{pix}} \int_{-dx_{pix}/2}^{dx_{pix}/2} x^2 = \frac{1}{12} dx_{pix}^2, \tag{21}$$

so that

$$\sigma_\theta = 17.3 \text{ arcsec } \frac{dx_{pix}}{1'}. \tag{22}$$

Although this is rather large, if its coherence length is the pixel scale, then the effect is quite managble.

This error creates a B power spectrum:

$$\frac{l^2 \Delta C_l^B}{2\pi} = \frac{1}{2}\sigma_\theta^2 (l/l_p)^2 \int \frac{dl}{l} \frac{l^4 C_l^E}{(2\pi)}, \tag{23}$$

where l_p is the pixel scale if the pixelization error is uncorrelated from pixel to pixel. For the EBEX beam

$$\int \frac{dl}{l} \frac{l^4 C_l^E}{(2\pi)} = (0.9 \ \mu\text{K/arcmin})^2. \tag{24}$$

We can also estimate $l_p = \sqrt{2\pi}/dx_{pix} = 8617 \times (1'/dx_{pix})$. For this values of l_p the effect is much smaller than the pointing error contribution. This is so because even though σ_θ is similar (a bit smaller because of the 12 in equation (21)) the effect on the power spectrum scales as l_p^2 and l_p is more than ten times larger than the 500 assumed for the pointing.

Although this effect seems very manageable, it would become larger if the coherence is larger than the pixel scale. I don't think that this will be the case. In

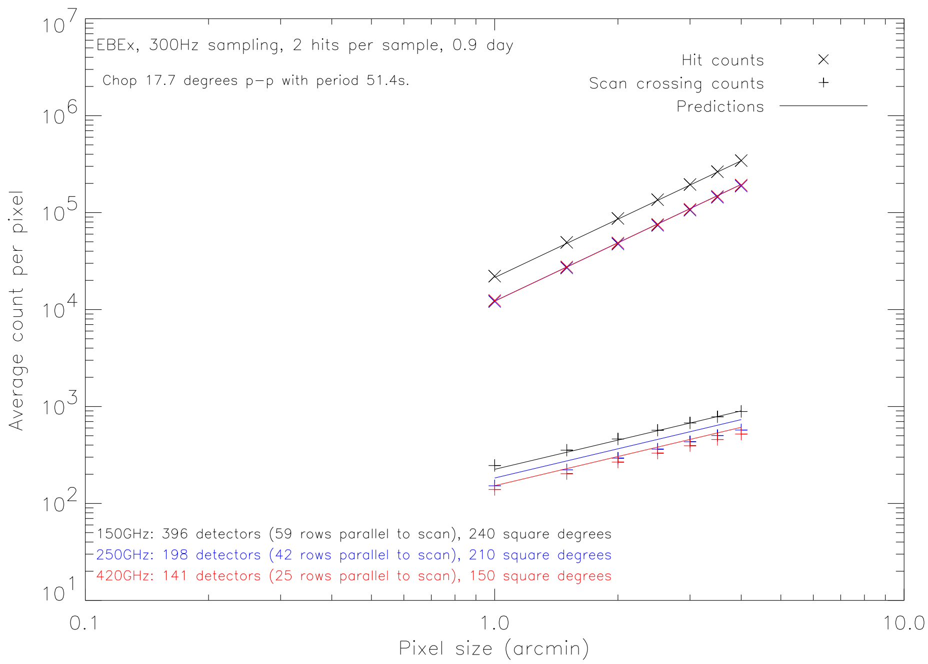

Figure 6: Scaling relations for the average hit and scan crossing counts per pixel. The numerical code (points) and scaling relations Eq. (18) and Eq. (19) (solid lines) are in good agreement.

any case, if it were so, going above the nearest grid point interpolation will make the pixelization error disappear.

A final comment on the coherence length of the pointing error. Along the direction of the scan the coherence is as large as the scan itself as the random walk is dominated by the first few modes. In the direction perpendicular to the scan however things are not that coherent. This is so because is a given scan one covers only a length $N_1 dx_{pix}$ which is smaller than the size of the focal plane. So to the extend that this is true, that only a small fraction of the size of the focal plane is covered as its sweeps across, the coherence in that direction will not be large. Once all the scans are put together the coherence will be some sort of average of the coherence in both directions. A more detailed analysis would be required to estimate this. It does not seem necessary given the smallness of the effect.

2.5 Derivation of the pointing noise formulas

Assume that the gondola is moving along the x axis and that measurements are take taken when the position is x_i at times t_i. At $i = 0$ and $i = N$ there is a measurement by the star camera. Between those the gyros are used. I will assume that the error in the star camera is Gaussian with variance σ_{sc}^2 and that the gyros measure difference in position with successive positions with variance σ_g^2.

After the scan the reconstructed pointing is $\hat{x}_i = x_i + \delta x_i$. Under the above assumptions the χ^2 is:

$$\chi^2 = \frac{\delta x_0^2}{\sigma_{sc}^2} + \sum_i \frac{(\hat{x}_{i+1} - \hat{x}_i - x_{i+1} + x_i)^2}{\sigma_{gi}^2} + \frac{\delta x_N^2}{\sigma_{sc}^2}. \tag{25}$$

We can now go to the continuous limit and use time t to label the position along the scan. We will also use that $\sigma_{gi}^2 = w_g dt_i$, where dt_i is the time between samples and w_g is a property of the gyro. We get:

$$\chi^2 = \frac{\delta x_0^2}{\sigma_{sc}^2} + w_g^{-1} \int dt (\frac{d\delta x}{dt})^2 + \frac{\delta x_N^2}{\sigma_{sc}^2}. \tag{26}$$

We now expand $\delta x(t)$ as,

$$\delta x(t) = \delta x_0 + t \frac{\delta x_N - \delta x_0}{t_{sc}} + \sum_n a_n \sin k_n t \quad , \quad k_n = \frac{n\pi}{t_{sc}}, \tag{27}$$

where t_{sc} is the time needed to cross between one side of the scan and the other. So that

$$\frac{d\delta x(t)}{dt} = [\frac{\delta x_N - \delta x_0}{t_{sc}} + \sum_n a_n k_n \cos k_n t]. \tag{28}$$

Plugging this into the χ^2 equation we get,

$$\chi^2 = \frac{\delta x_0^2}{\sigma_{sc}^2} + \frac{1}{w_g t_{sc}} [(\delta x_N - \delta x_0)^2 + \frac{1}{2} \sum_n (n\pi)^2 a_n^2] + \frac{\delta x_N^2}{\sigma_{sc}^2}. \tag{29}$$

This explains why we choose this particular expansion, it makes all the a_n independent. The variance of the different random variables can be read-off from this expression.