GWE Spring 2010 - Short Problems

1: Consider a thin uniform and rigid rod of mass m and length L. A small ball of mass M is attached to one end of the rod. The other end of the rod is suspended from the ceiling and the system is free to oscillate about the suspension point without friction. Compute the period of the small oscillations (in a plane) of this system. Verify that you obtain the expected result when $M \gg m$.

2: Consider a semi-infinite one dimensional potential well. The potential is infinite at $x = 0$, it is zero for $0 < x < a$, and it has the finite value $V_0 > 0$ for all $x > a$. Compute the minimum value of a for which such a potential can confine a particle of mass m.

3: The electron has mass $m_e = 0.511\,\mathrm{MeV}/c^2$. The top quark has mass $m_t = 173\,\mathrm{GeV}/c^2$. A machine produces a beam of electrons, of energy E_1 each. A second machine produces a beam of positrons, of energy E_2 each. The two beams are made to collide head on. A total amount of energy can be given to the electrons and positrons beams, in whatever ratio; namely $E_1 = x\,E,\ E_2 = (1-x)\,E$. (i) For any value of x, compute the threshold energy E that allows the production of pairs of real top and anti-top quarks when one electron collides with a positron (you may disregard m_e when compared to m_t). (ii) Which choice of x gives the smallest value of E ?

GWE Spring 2010 - Long Problems

1: Two identical objects A and B, of mass m each, are connected by a spring, of spring constant k. At $t = 0$ the two objects are at rest, and the spring is in its equilibrium position. For $t > 0$, the object A is subject to an external force $F_{\rm ext} = F \cos(\omega t)$, with F and ω constant, as shown in Figure 1. Compute the motion of the object B for any $t \geq 0$. Neglect all friction.

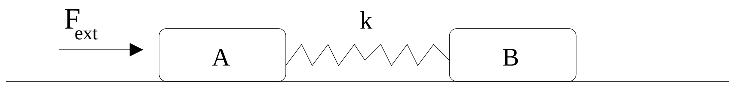

Figure 1: Long problem 1

2: The neutron has the magnetic dipole moment

$$\vec{\mu} = \gamma \vec{S} \tag{1}$$

where γ is a constant, and $\vec{S}$ is the spin of the neutron.

A nonrelativistic neutron with momentum $\vec{k}$ is moving in a uniform and constant magnetic field. The interaction between the neutron magnetic dipole moment and the magnetic field gives a term in the hamiltonian $\hat{H}$ of this system. (i) Write down the complete hamiltonian $\hat{H}$.

(ii) Assume that the magnetic field is $\vec{B} = (0, 0, B_z)$. What are the possible energies for the neutron? What are the corresponding normalized wave functions? (to get the normalization, require that there is a probability one that the neutron is at some place in a large volume L^3).

(iii) Answer the same questions as in part (ii), in the case of a magnetic field $\vec{B} = (B_x, 0, B_z) \equiv |\vec{B}| (\sin\theta, 0, \cos\theta)$.

(iv) Assume now that B_x is very small, and can be treated as a perturbation on the problem solved at point (ii), where B_x was taken to vanish. Starting from the unperturbed solutions obtained in (ii), compute the possible energies of the neutron to first order in B_x. Compare with the exact energies obtained in (iii).

3: Let $Z_1(m)$ be the partition function of a single (quantum) particle of mass m in a volume L^3, at the temperature T.

(i) Consider a system of two such particles, assuming that they do not interact. Denote by $Z_{2,\text{dist}}(m)$ the partition function of the system assuming that the two particles are distinguishable. Express this quantity in terms of $Z_1(m)$.

(ii) Assume now that the two particles are indistinguishable spin zero bosons. Denote by $Z_{2,\text{bose}}(m)$ the partition function for this system. Express this quantity in terms of $Z_1(m)$ and $Z_1(m/2)$.

(iii) Comparing the cases (i) and (ii), calculate (to lowest order in the quantum effects) the correction to the expectation value of the energy of the two particle system due to Bose statistics. In which regime is the correction negligible?

4: Consider a system of two particles, with identical masses, orbiting in a circle around their center of mass. (i) Show that the gravitational potential energy of the system is -2 times the total kinetic energy.

(ii) This relation is true, on average, for any system of particles held together by their mutual gravitational attraction: $\bar{U}_{\text{potential}} = -2\,\bar{U}_{\text{kinetic}}$, where $\bar{U}$'s are the total amount of potential and kinetic energies, averaged over some sufficiently long time. Suppose that you add a small amount of energy to such system, and then you wait until it equilibrates. Will the particles in the system, on average, move faster, or more slowly? Explain.

(iii) Compute the potential energy for a uniform spherical distribution of particles of radius R and total mass M.

(iv) Assume that a star can be modeled by an ideal gas of particles obeying classical statistics, at the same temperature T, which interact among themselves only gravitationally. Estimate the temperature of a star of mass $M = 2 \times 10^{30}\,\text{Kg}$ and radius $R = 7 \times 10^{8}\,\text{m}$. Assume for simplicity that the star contains only protons and electrons.

5: Consider a uniform infinitely long cylindrical wire, of cross section area A, with a current I flowing through it. Consider a charged object, of charge $q > 0$, moving parallel to the wire, with speed v. The object is outside the wire, at the distance d from it ($d \gg \sqrt{A}$). The wire is neutral, and the object moves in the direction opposite to the flow of the current in the wire.

(i) Compute the magnitude and direction of the magnetic force $\vec{F}$ acting on the charged object.

(ii) Assume the following idealized situation for the wire: the wire is made of only protons and electrons, uniformly distributed within it. The proton and electrons have the same number density n (n has dimension of inverse volume). The protons are at rest, while all the electrons move with the same velocity $\vec{v}$. Assume that this velocity is equal (both in magnitude and direction) to that of the outside object. Express the current I in terms of v (and of any other relevant parameter), and insert this expression in the formula for the magnetic force computed in (i).

All the above statements are made by an observer $\mathcal{O}$ at rest with respect to the wire. Consider now the same situation in the rest-frame of the outside charged object.

(iii) Does the object experience a magnetic force in this frame?

(iv) Compute the number densities of protons (n'_+) and electrons (n'_-) inside the wire in this frame (hint 1: the electric charge of any individual particle is the same in both frames; hint 2: notice that, due to the symmetry of the problem, there is a simple relation between the ratio n'_+/n and the ratio n'_-/n).

(v) Compute the linear charge density of the wire in this frame (charge per unit length along the wire). Compute the force $\vec{F}'$ acting on the outside object in this frame.

(vi) Show that the resulting ratio $\vec{F}'/\vec{F}$ is only function of the γ factor between the two frames, and of no other parameters. Show that this result is the one you would have expected, given that $\vec{F} = \Delta\vec{p}/\Delta t$, $\vec{F}' = \Delta\vec{p}'/\Delta t'$, and how $\Delta\vec{p}$ and Δt are related to $\Delta\vec{p}'$ and $\Delta t'$.

6: A particle of mass m is confined to slide on the surface of an "upside down" cone with semi-angle α, as shown in Figure 2, and is subject to the constant gravitational field of the Earth surface. The axis of the cone is on the $z-$axis. Neglect any form of friction for points (i) and (ii).

(i) Write down the Lagrangian for this particle, using the coordinates r and θ, defined by $x = r\,\cos\theta$ and $y = r\,\sin\theta$ (notice that r and θ completely specify the position of the particle on the surface of the cone). Write down the Euler-Lagrange equations, obtained from this Lagrangian, that describe the motion of the particle.

(ii) For appropriate speed $|\vec{v}|$, the particle can move on a horizontal - and therefore circular - trajectory with $z = \bar{z} =$ constant. Write down the relation between $\bar{z}$ and the speed. Write down the total energy for the particle in this motion.

(iii) For this part only, assume that the cone is filled by some viscous medium, so that the particle is subject to a dragging force $\vec{F}_{\text{drag}} = -\,b\,\vec{v}$, where b is constant and $\vec{v}$ is the velocity of the particle. Assuming that the particle is initially (at $t = 0$) on a circular horizontal orbit, with height $\bar{z}_0$, and that the effect of the drag is small, so that the orbits of the particle can be approximated as circular at all times (with a very slowly decreasing radius, due to the drag), compute the time evolution of the height of the particle $\bar{z}\,(t)$.

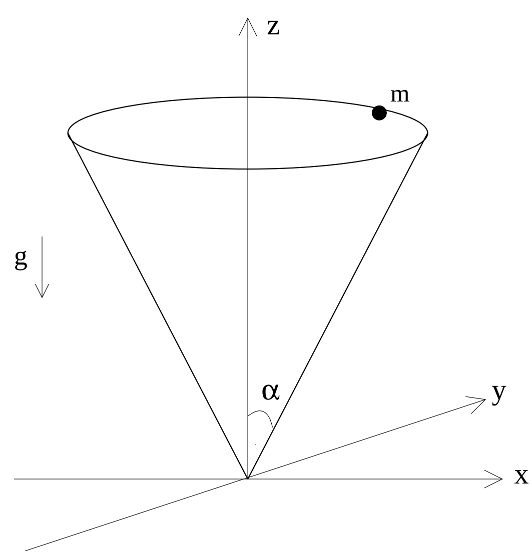

Figure 2: Long problem 6

4: A very long linear solenoid is made of n circular loops per unit length. The area of each loop is A. The current in this solenoid is increased linearly with time, $I = \alpha\, t$, where α is a constant. (i) What is the magnetic field inside this solenoid?

The solenoid is placed perpendicularly to a planar circuit, as shown in Figure 1 (the solenoid extends both inside and outside the page, for a distance much greater than the dimensions of the circuit; the arrows on the figure show the direction of the current in the loops of the solenoid). The circuit shown in the figure consists of two resistors, of resistances R_1 and R_2, and two voltmeters. The internal resistances of the two voltmeters are much greater than R_1 and R_2. (ii) What are the magnitudes of the potential differences V_1 and V_2 measured by the two voltmeters?

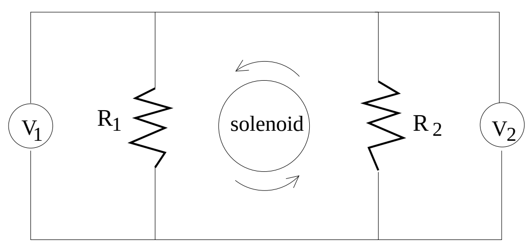

Figure 1: Short problem 4

5: Consider two identical and coaxial superconducting loops. Each loop has self-inductance L. Initially, the two loops are very far apart from each other, and a current I flows in each of them; the currents in the two loops have the same direction. Starting from this initial configuration, the two loops are then "translated" one on the top of the other, and superimposed (you can assume that they do not touch, although their distance becomes negligible). What is the final current in each of them? What are the initial and final energies of the system?

6: Two parallel perfectly black planes are in a vacuum, and are kept at constant and different temperatures T_1 and T_2. Denote by Φ the heat flux between these two planes. If a third perfectly black plane is inserted between these two planes, the system reaches a new steady state, for which the flux between the two external plates is $\Phi' \neq \Phi$. Compute the ratio Φ'/Φ.

7: A nonrelativistic particle of mass m and electric charge q is in the ground state of a one-dimensional simple harmonic oscillator potential $V(x) = m\,\omega^2\,x^2/2$. Recall that the normalized wavefunction for this state is

$$\psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\,\omega\,x^2}{2\hbar}\right) \tag{1}$$

At some moment, a uniform electric field in the x direction is switched on very quickly (i.e., on a timescale which can be regarded as instantaneous for this problem), and is then kept constant. (i) Show that the new (i.e., with the electric field switched on) potential to which the particle is subject is of the simple harmonic oscillator type. (ii) Compute the probability that the particle is found in the ground state of this new potential.

8: The latent heat of melting for ordinary ice is 334 J/g. Use this and your own experience on how the volumes of ice and water differ to determine the sign and estimate the slope of the melting curve for water in the $p - T$ (pressure and temperature) plane.

9: An engine with 1 mol of an ideal gas starts at $V_1 = 26.9$ liters and performs a cycle consisting of four steps:

1. Heating at constant pressure to twice its initial volume, $V_2 = 2\,V_1$
2. Isothermal expansion at T_2 to $V_3 = 3\,V_1$
3. Cooling at constant volume to $T_1 = 250$ K
4. Isothermal compression to its original volume V_1

Assume that the molar heat capacity at constant volume for this gas is $C_V = 21$ J/K. (i) Calculate the P, V, T (pressure, volume, temperature) points, and draw the engine cycle on a P-V diagram. (ii) Determine the efficiency of this engine.

10: A particle of mass M is initially moving along the $x-$axis, with constant speed v (as measured in the laboratory frame), which can vary from 0 to near the speed of light. This particle decays into two identical particles of mass m, with isotropic probability in its rest frame. In the following, unprimed quantities are in the laboratory frame, while primed quantities are in the rest frame of the initial particle. Choose the x and x' axis of these two frames to coincide. Denote by $\vec{p}$ the momentum of one of the decay products. Choose the axes such that $p_z = p'_z = 0$. Denote by θ the angle in the laboratory frame between the $x-$axis and the velocity of this decay product ($\theta = 0$ if the decay product moves in the direction of the initial particle). The corresponding angle θ' in the rest frame of the initial particle is shown in Figure 2. (i) For a given θ', compute p'_x, p'_y, p_x, p_y, and determine the relation between θ and θ'. (ii) Consider a beam of many such initial particles, all moving along the same straight line with velocity v. Consider the value of v for which, in the laboratory frame, half of the decay products are emitted inside a cone forming an angle $\theta \leq \theta_0$ with the direction of the initial beam. Find the relation between θ_0 and v, as v varies from 0 to the speed of light.

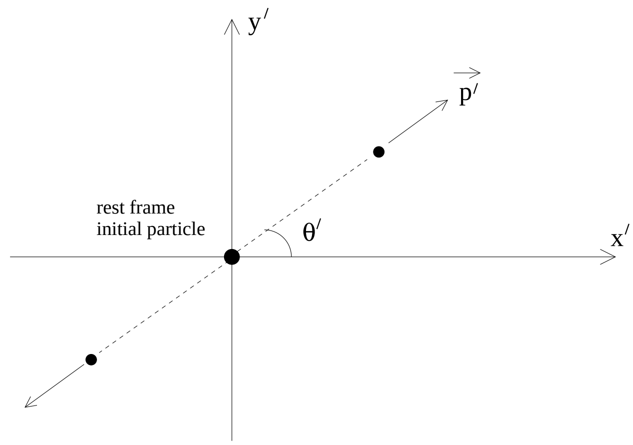

Figure 2: Short problem 10

11: A cook has a spherically shaped soup spoon. On looking into the concave side he sees his inverted image 4 cm from the bottom of the spoon (see Figure 3). Without changing his distance to the spoon, he turns it over, and sees an erect image of himself 3 cm from the bottom of the spoon. What is the radius of curvature of the spoon?

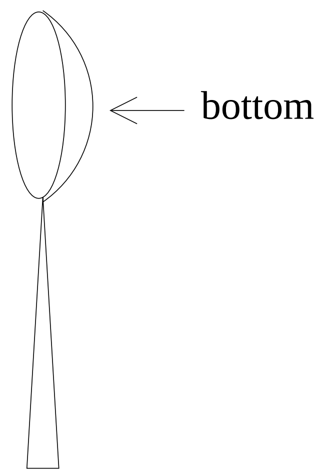

Figure 3: Short problem 11

12: In one day in 1987, the IMB detector observed 8 neutrino interactions. The normal background interaction rate in the detector was two a day. (i) What is the probability of eight background events being detected in one day? (ii) In fact, all those neutrinos occurred in a 10 second period. What is the probability that all those events were due to a background fluctuation?