Problem 1

If dust particles are sufficiently small, they will be swept out of the Solar System by radiation pressure from the Sun. Show that this is true independently of the distance to the Sun, and calculate the critical radius for this to happen to a totally absorbing dust particle of density $\rho = 0.5$ g/cm^3. The solar energy flux at the Earth is $S_{Sun} = 1.4\ k\text{W/m}^2$.

Problem 2

A ball is thrown vertically upward from the ground at $y = 0$ with an initial speed v_0. In addition to the constant downward acceleration g exerted by gravity, the ball is subjected to a drag force resulting in an acceleration of magnitude γv^2 in the direction opposite to v. Taking γ to be constant, to what maximum height H will the ball rise? Check your answer by verifying that in the limit $\gamma \to 0$ you recover the well-known result for H in the absence of a drag force. Hint: Express the acceleration $\dot{v}$ as a function of y rather than t in the equation of motion.

Problem 3

A one-dimensional harmonic oscillator of frequency ω is prepared at time t = 0 in the following linear combination of the ground and first excited state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Show that the expectation value of the position operator oscillates when t > 0 and give the frequency and amplitude of the oscillation.

Note: For the calculation of the expectation value of $\hat{x}$, recall that the first excited state results from the application of the creation operator $a^\dagger$ to the ground-state, with

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\ \hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$$

Problem 4

When a charged particle passes through a transparent medium at a speed greater than the phase velocity of light in that medium, the so-called Cherenkov radiation is emitted in the forward cone co-axial with the path of the particle. Now consider the following situation: Pions (mass $m_\pi c^2 = 140$ MeV) and muons (mass $m_\mu c^2 = 106$ MeV) with the same momentum $|\mathbf{p}|$ = 140 MeV/c travel through a transparent material. Find the range of index of refraction of this material over which the muons alone will emit Cherenkov light.

Problem 5

A copper wire of length L = 1 km is connected across a 6-volt battery. Take the resistivity and number density of conduction electrons of copper to be respectively $\rho = 2\times10^{-8}$ ohm·meter and $n = 8\times10^{28}\ m^{-3}$. How long does it take for a conduction electron to drift around the circuit?

Problem 6

The temperature of 10 liters of nitrogen gas (N_2) held in a rigid container and originally at 0 °C and atmospheric pressure is raised to 100 °C by placing it in contact with a very large reservoir at 100 °C. What are the resulting changes in entropy in J/K (a) of the nitrogen gas and (b) of the gas and reservoir together.

Problem 7

A positron e^+ moving along the x-axis with kinetic energy 1.00 MeV collides with an electron e^- at rest, resulting in the creation of a pair of photons in the process $e^+e^- \to \gamma\gamma$. Positrons and electrons are each other's antiparticles and have the same mass $m_e c^2 = 0.51$ MeV. In this case, both photons come out with the same energy E_γ. Find the energy and momentum E_γ and p_γ of each photon (in MeV and MeV/c respectively) as well as the angles with respect to the x-axis at which the photons are emitted.

Problem 8

Consider a particle of mass m moving in the following one-dimensional potential (the "half-harmonic oscillator")

$$V(x) = \begin{cases} \infty, & x < 0 \\ \dfrac{1}{2} m\omega^2 x^2, & x \geq 0 \end{cases}$$

What are the energies and eigenstates for this system? In particular, what are the ground state and the first excited state energies?

Note that you are not expected to solve the Schrödinger equation here – rather, state your answers and justify them. Recall that the eigenstates of the usual simple harmonic oscillator (defined over the entire interval $-\infty < x < \infty$) are given by

$$\psi_n(x) = C_n\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)\, e^{-m\omega x^2/(2\hbar)}$$

Here, C_n is a normalization constant and the functions $H_n(x)$ are Hermite polynomials of order n. Hint: Consider what boundary condition must hold at x = 0.

Problem 9

An electric charge distribution produces an electric field

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\left(1 - e^{-r/a}\right)\frac{\hat{\boldsymbol{r}}}{r^2}$$

where q is some charge and a is a constant with units of length. Find the net charge contained within the radius $r = a$.

Problem 10

Nuclear reactors produce a lot of electron antineutrinos as a result of the rapid beta decay of neutron-rich fission products. Each fission reaction produces about 200 MeV of energy and an average of 2.5 antineutrinos that are so weakly interacting that their mean free path in the material of a suitable detector is about 5×10^{17} meters. Nevertheless, experiments are underway that do detect these antineutrinos!

Estimate how many antineutrinos will be detected per day in the above detector (assumed to have an efficiency of 30%) in the shape of a cube of side L = 1 meter that is placed a distance D = 25 meters away from the core of a reactor producing thermal energy at the rate of 1 GW.

Problem 1

If the Solar System were immersed in a uniformly dense spherical cloud of dark matter, objects within would experience a net gravitational potential from both the Sun and the dark matter of the form

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2$$

(a) Using plane polar coordinates (r, φ), write down the Lagrangian for an object of mass m moving in this potential and derive the resulting equations of motion, identifying all conserved quantities. In particular, write down the equivalent one-dimensional equation of motion for r.

(b) Find the condition under which the object will move in a circular orbit of radius R. You need not solve for R !

(c) Find the frequency ω for small radial oscillations δr of the object in a nearly circular orbit in terms of R and the potential parameters k and b.

Problem 2

Consider two observable (Hermitian) operators $\hat{A}$ and $\hat{B}$ in a three-dimensional Hilbert space. In the basis

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the observables are represented by the matrices

$$A = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = b\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

(a) Show that the observables $\hat{A}$ and $\hat{B}$ are not compatible, that is they do not admit a common set of eigenstates.

(b) Find the eigenvalues and corresponding normalized eigenstates of the observables $\hat{A}$ and $\hat{B}$.

(c) What are the possible outcomes and the corresponding probabilities of separate (independent) measurements of $\hat{A}$ and $\hat{B}$ in the state $|\chi\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$.

(d) What are the possible outcomes and the corresponding probabilities of a measurement of $\hat{B}$ that follows a measurement of $\hat{A}$ if the system is initially in the state $|\chi\rangle$ of part (c).

(e) Same question as in (d), except now when a measurement of $\hat{A}$ follows a measurement of $\hat{B}$.

Problem 3

Three spin-1/2 atoms with spin operator $\boldsymbol{s}_i$ (i=1,2,3) sit at the corners of an equilateral triangle with their mutual interactions described by the Hamiltonian

$$H = \frac{\lambda}{3}(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 + \boldsymbol{s}_1 \cdot \boldsymbol{s}_3 + \boldsymbol{s}_2 \cdot \boldsymbol{s}_3)$$

(a) List the possible values of the total spin S for this system as well as the corresponding degeneracies for each state of given S.

(b) Write the Hamiltonian in terms of the total spin $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3$ and find the corresponding eigenvalues of the energy of the system.

(c) Write down the canonical partition function Z at finite temperature T and obtain the internal energy of this system.

Problem 4

In the x'-y'plane of an inertial frame S', a rod of proper length L_0 is at rest, inclined at an angle θ_0 with respect to the x' axis. The rod is moving with velocity $\mathrm{v} = \beta\, c$ with respect to an observer at rest in inertial frame S whose x-axis points in the direction of the x' axis of S'.

(a) Determine the length L of the rod as measured by the stationary observer in S in terms of L_0 and θ_0. Verify that your expression reduces to the expected results when θ_0 takes on the values 0° and 90°.

(b) Determine the angle θ the rod makes with respect to the x axis in S.

(c) Numerically, what are L and θ if $L_0 = 1\, m,\ \theta_0 = 45°$ and $\beta = 0.6$?

Problem 5

A circular loop of radius a lies in the x-y plane with its center at the origin of coordinates. It carries a uniformly distributed total charge Q so that its charge density (per unit volume) can be expressed as $\rho(\boldsymbol{r}) = \frac{Q}{2\pi a}\,\delta(s-a)\,\delta(z)$, where $\boldsymbol{r} = s\,\hat{\boldsymbol{s}} + z\hat{\boldsymbol{z}}$ in cylindrical coordinates with $\hat{\boldsymbol{s}} = \hat{\boldsymbol{x}}\cos\varphi + \hat{\boldsymbol{y}}\sin\varphi$.

(a) Calculate the resulting scalar potential $\phi(\boldsymbol{r})$ exactly at any point $\boldsymbol{r} = (0,0,z)$ on the symmetry axis of the loop.

For distances $r \gg a$, the scalar potential can be expanded in multipoles,

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2r^5}\,Q_{ij}r_ir_j + \cdots\right\}$$

(b) Expand the exact expression for $\phi(0,0,z)$ in powers of $1/|z|$ and identify the relevant components of the dipole (p_i) and quadrupole (Q_{ij}) moments.

(c) Verify and extend your findings in part (b) by calculating all the components of the dipole and quadrupole moments of the charged loop.

It may be helpful to recall here the expression for the quadrupole moment,

$$Q_{ij} = \int d^3r\;\rho(\boldsymbol{r})\,(3r_ir_j - r^2\delta_{ij})$$