Graduate Written Examination

Spring 2014 – Part I

Thursday, January 16th, 2014 – 9:00am to 1:00pm

University of Minnesota
School of Physics and Astronomy

Examination Instructions

Part 1 of this exam consists of 10 problems of equal weight. You will be graded on all 10 problems. For each problem provide as complete a solution as you can.

This is a closed-book examination. You may use a calculator. A list of some physical constants and properties that you may require is included. **Please take a moment to review its contents before starting the examination.**

Please put your assigned **CODE NUMBER, *not your name or student ID*,** in the **UPPER RIGHT-HAND CORNER** of each piece of paper that you submit, along with the relevant **PROBLEM NUMBER** in the **UPPER LEFT-HAND CORNER**.

BEGIN EACH PROBLEM ON A FRESH SHEET OF PAPER, so that no sheet contains work for more than one problem.

USE ONLY ONE SIDE of the paper; if you require more than one sheet, be sure to indicate, "page 1", "page 2", etc., under the problem number already entered on the sheet.

Once completed, all your work should be put in the manila envelope provided, **IN ORDER** of the problem numbers

Constants	Symbols	Values
Speed of light in vacuum	c	3.00×10^8 m/s
Elementary charge	e	1.60×10^{-19} C
Electron rest mass	m_e	9.11×10^{-31} kg
Electron rest mass energy	$m_e c^2$	0.511 MeV
Permeability constant	μ_o	1.26×10^{-6} H/m
Permeability constant $/4\pi$	$\mu_o/4\pi$	10^{-7} H/m
Vacuum Dielectric Constant	$1/4\pi\epsilon_0$	8.99×10^9 N m^2/C^2
Proton rest mass	m_p	1.67×10^{-27} kg
Proton rest mass energy	$m_p c^2$	938 MeV
Neutron rest mass	m_n	1.68×10^{-27} kg
Neutron rest mass energy	$m_n c^2$	940 MeV
Planck constant	h	6.63×10^{-34} J·s
Gravitational constant	G	6.67×10^{-11} m^3/s^2·kg
Molar gas constant	R	8.31 J/mol·K
Avogadro constant	N_A	6.02×10^{23} /mol
Boltzmann constant	k_B	1.38×10^{-23} J/K
Molar volume of ideal gas at STP	V_m	2.24×10^{-2} m^3/mol
Earth radius		6.37×10^6 m
Earth-Sun distance		1.50×10^{11} m

Stirling's Approximation	$\ln(N!) = N\ln(N) - N +$(small corrections)
	$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

Problem 1

Consider the one-dimensional motion at positive displacement x of a particle subject to a force $F = -\frac{b_1}{x^3}$ where b_1 is a positive constant.

1. Find the escape velocity from position x_0.
2. Suppose the force was $F = -\frac{b_2}{x}$, (b_2 is also positive). Show that the escape velocity is infinite.

Problem 2

A classical particle of mass m moves in a closed orbit in the gravitational field of a mass $M >> m$; M is at the origin and the potential energy is $V = -k/r$. Besides energy and angular momentum there is another conserved quantity, the Laplace-Runge-Lenz vector $\mathbf{A}$

$$\mathbf{A} \ = \mathbf{p} \times \ \mathbf{L} - \frac{mk}{r}\ \mathbf{r},$$

where $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{L}$ denote the linear momentum, position, and angular momentum of the particle, respectively.

1. In one or two lines argue why $\mathbf{A}$ is in the plane of the orbit.
2. The magnitude of the vector $\mathbf{A}$ is related to the eccentricity ϵ of the orbit. Recalling that when the origin coincides with one of the foci the equation of an ellipse can be written as
$$\frac{1}{r} = C(1 + \epsilon \cos\theta),$$
where θ is the azimuthal angle measured relative to $\mathbf{A}$ ($\mathbf{A}\cdot\mathbf{r}= Ar\cos\theta$) and C is a constant, relate the magnitude A to the eccentricity ϵ, the mass m, and k.

Problem 3

Find the magnetic flux through a square loop due to a current I in a long straight wire. The loop is coplanar with the wire and has two sides parallel to it. The loop has side length a and its side nearest to the wire is a distance b from the wire.

Problem 4

A system is described by the wave function $\Psi = A\text{cos}^2\phi$, where A is a normalization constant and ϕ is the azimuthal angle. You measure the angular momentum of this system along the z axis. Compute which results you can obtain from such a measurement, and their probabilities.

Problem 5

Derive the average momentum $\langle \hat{p} \rangle$ for a packet with a normalizable wave function of the form

$$\Psi(x) = C\,\phi(x)\exp(ikx)\,,$$

where C is a normalization constant and $\phi(x)$ is a real function.

Problem 6

Galaxy A moves away from our galaxy at a speed of 0.6c. Galaxy B moves away at 0.7c with a trajectory that is 45 degrees to A. Write down the transformation of space and time differential coordinates (Δt, $\Delta x, \Delta y$) between two objects moving at constant velocity v_0 relative to each other (orient the axes as you see fit), derive the velocity transformation, and express the velocity of B as observed by the civilization in A.

Problem 7

A circular storage ring with an orbit diameter of 10 meters stores protons with 1 GeV kinetic energy.

1. How long do they take to complete one orbit?
2. What magnetic field strength is needed to constrain them in orbit?

Problem 8

Consider a planar square lattice with N classical spins at each site i represented by the unit vector $\mathbf{S}_i$. Each spin is restricted to point along only four directions: $\pm\hat{\mathbf{x}}$ and $\pm\hat{\mathbf{y}}$. Each spin interacts only with its nearest neighbors according to the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$

where $J > 0$ is the exchange interaction and $\langle ij \rangle$ denotes a pair of nearest neighbor sites. Since we are ultimately interested in the thermodynamic limit $N \to \infty$ the effect of the boundary is negligible.

By computing the internal energy of the ferromagnetic ordered state (in which all spins are perfectly parallel to each other) and the entropy of the paramagnetic state (in which the spins are completely disordered), and by considering the free energy of the system give an estimate for the temperature at which the system undergoes a phase transition from the ferromagnetic to the paramagnetic state in the limit $N \to \infty$.

Problem 9

Two protons are at large distance from each other. Initially one proton is at rest and the other is moving head-on toward the first with kinetic energy $\mathcal{E}$. The motion is nonrelativistic, $v \ll c$. Find the minimal distance r_{min} between the protons. At what energy will the minimum distance between the protons be $r_{\text{min}} = 10^{-13}$ cm?

Problem 10

Mirrors 1 and 4 in Figure 1 are 'half-silvered' so half the intensity of light incident upon them is transmitted and half is reflected. Ignore multiple reflections at all the mirrors and assume that other reflections and attenuations are negligible.

1. Find the dependence of the intensity of transmission of the device on L, the indices of refraction n_1 and n_2, and the wavelength λ of the incident monochromatic, phase coherent light.

2. You are provided with a crude light detector that can only measure the maximum and minimum of intensity with any confidence (but not intermediate light levels). You are starting with both chambers evacuated and you have a pressure gauge. Show how to determine the index of refraction of a gas as a function of pressure by monitoring the transmission intensity as gas is added to one of the tubes.

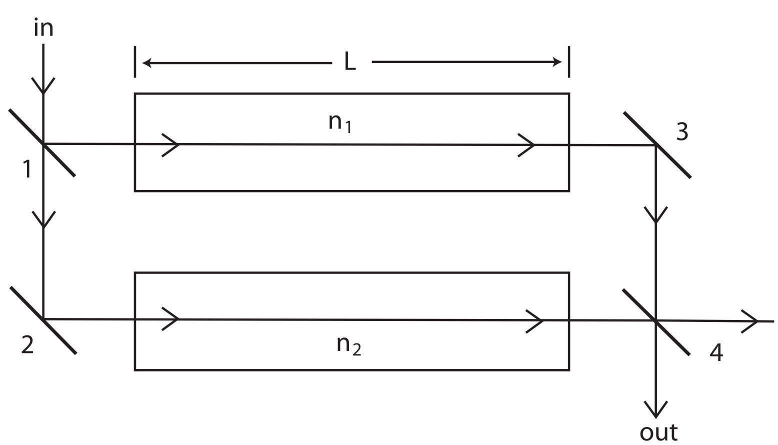

Figure 1: Problem 10

Graduate Written Examination

Spring 2014 – Part II

Friday, January 17th, 2014 – 9:00am to 1:00pm

University of Minnesota

School of Physics and Astronomy

Examination Instructions

Part 2 of this exam consists of 5 problems of equal weight. You will be graded on all 5 problems. For each problem provide as complete a solution as you can.

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Problem 1

A bead of mass m is constrained to move on a frictionless hoop (see Figure 1). The hoop of radius R is forced to rotate about the vertical diameter with angular speed ω. The position of the bead is specified by coordinate θ. Write the Lagrangian for the system and derive the differential equation of motion for θ. Include the gravitational force in your analysis. Determine the equilibrium solutions for this problem, that is, the values of constant θ that solve the differential equation of motion. For each solution, discuss the conditions under which this solution exists and indicate whether it is a stable or an unstable equilibrium solution.

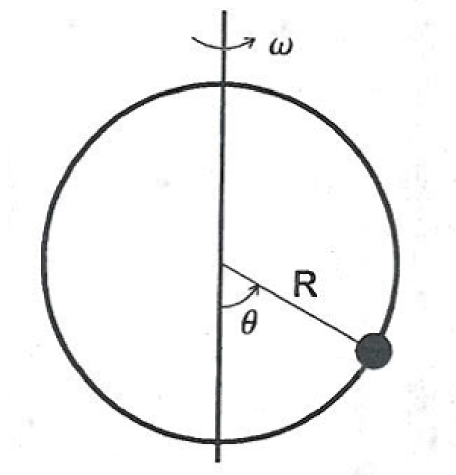

Figure 1: Problem 1

Problem 2

A charge q is uniformly distributed throughout a spherical volume of radius a. Surrounding and concentric with this charge distribution is an uncharged, conducting spherical shell of inner radius b and outer radius c ($c > b > a$). The space between the charge distribution and the conducting shell is filled with a linear isotropic, homogeneous dielectric, having dielectric constant κ. Find the polarization vector $\vec{P}$ in the dielectric, and find the electrostatic field $\vec{E}$ and potential V everywhere. Take V to approach zero at large distances from this configuration. Make two sketches highlighting the main features of the functional dependence of $\vec{E}$ and V on the radial coordinate.

Problem 3

An ideal gas engine is working in a reversible Joule cycle shown in the T vs. S diagram (figure 2). The gas is monatomic and has n moles.

1. Draw the corresponding P-V diagram for the cycle.
2. Express the work done in terms of temperatures T_1, T_2, T_3, and T_4.
3. Find the efficiency of the engine in terms of these temperatures.

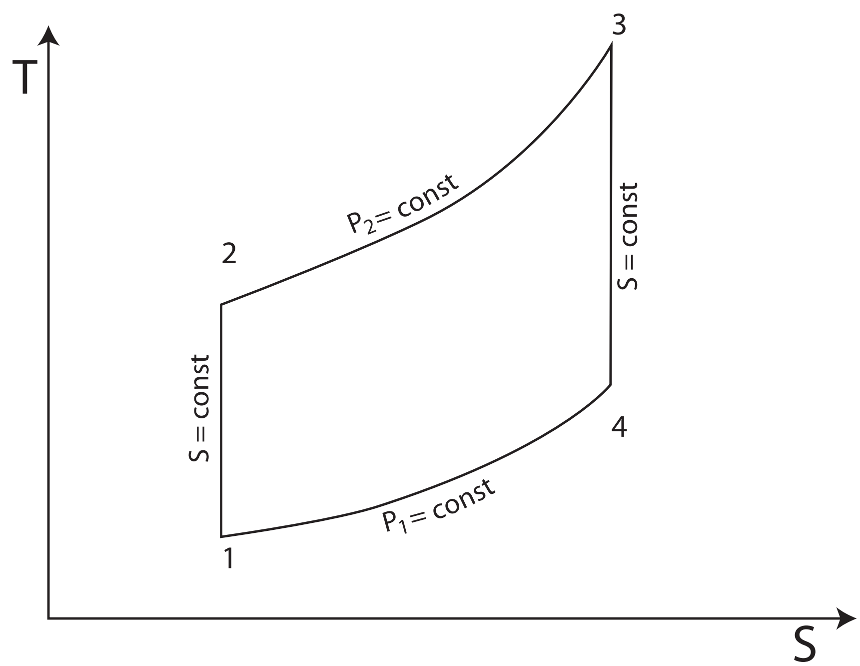

Figure 2: Problem 3

Problem 4

A particle is in an infinitely deep one-dimensional potential well of the width a located at $0 \leqslant x \leqslant a$

$$\begin{aligned} V(x) &= 0\,, \qquad 0 \leqslant x \leqslant a\,, \\ V(x) &= \infty\,, \quad x \leqslant 0\,, \quad x \geqslant a\,. \end{aligned}$$

1. Find the normalized wave functions that describe its energy states.
2. Find the first order corrections ΔE_n to the energy levels for a perturbation of the form

$$\begin{aligned} \Delta V(x) &= V_0\,\frac{2x}{a}\,, \qquad 0 \leqslant x \leqslant \frac{a}{2}\,, \\ \Delta V(x) &= V_0\left(2 - \frac{2x}{a}\right), \qquad \frac{a}{2} \leqslant x \leqslant a\,. \end{aligned}$$

(Hint: to solve integrals of the form $\int x \cos kx\, dx$ you may use the technique of differentiating with respect to a parameter, as in $\int x \cos kx\, dx = \frac{d}{dk} \int \sin kx\, dx$.)

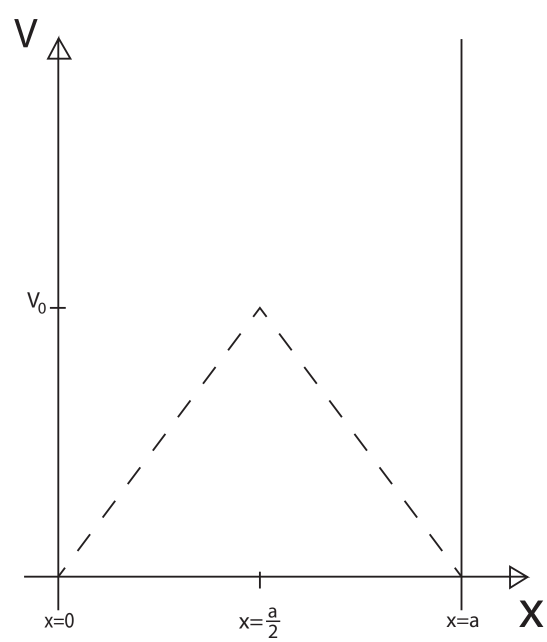

Figure 3: Problem 4

Problem 5

It has been proposed to create a terrestrial source of high energy photons by having low energy laser photons collide with accelerator-generated high energy electrons. Consider a laser photon with energy $\hbar\omega = 1$ eV that collides head-to-head with an electron and reflects backward. The energy of the electron is $E = 50$ GeV ($= 50 \times 10^9$ eV). Find the energy of the photon after the collision.

[Hints: 1. note that you cannot neglect the electron mass because it is much larger than the photon energy; 2. to simplify calculations you may want to transform to (and back from) the center of mass frame. A different approach (with the same goal) is to make use of $(E + pc)(E - pc) = E^2 - p^2c^2 = m^2c^4$].