University of Minnesota
School of Physics and Astronomy

GRADUATE WRITTEN EXAMINATION

Spring 2015 – PART I

Thursday, January 15th, 2015 – 9:00 am to 1:00 pm

Part 1 of this exam consists of 10 problems of equal weight. You will be graded on all 10 problems.

This is a closed-book examination. You may use a calculator. A list of some physical constants and properties that you may require is included. Please take a moment to review its contents before starting the examination.

Please put your assigned **CODE NUMBER (not your name or student ID)** in the **UPPER RIGHT-HAND CORNER** of each piece of paper that you submit, along with the relevant problem number in the **UPPER LEFT-HAND CORNER.**

BEGIN EACH PROBLEM ON A FRESH SHEET OF PAPER, so that no sheet contains work for more than one problem.

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Once completed, all your work should be put in the manila envelope provided, **IN ORDER** of the problem numbers.

Constants	Symbols	values
Speed of light in vacuum	c	3.00×10^{8} m/s
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Electron rest mass energy	m_ec^2	0.511 MeV
Permeability constant	μ_o	1.26×10^{-6} H/m
Permeability constant/4π	$\mu_o/4\pi$	10^{-7} H/m
Proton rest mass	m_p	1.67×10^{-27} kg
Proton rest mass energy	m_pc^2	938 MeV
Neutron rest mass	m_n	1.68×10^{-27} kg
Neutron rest mass energy	m_nc^2	940 MeV
Planck constant	h	6.63×10^{-34} J–s
Gravitational constant	G	6.67×10^{-11} m^3/s^2–kg
Molar gas constant	R	8.31 J/mol–K
Avogadro constant	N_A	6.02×10^{23} /mol
Boltzmann constant	k_B	1.38×10^{-23} J/K
Molar volume of ideal gas at STP	V_m	2.24×10^{-2} m^3/mol
Earth radius	R_E	6.38×10^{6} m
Earth's mass	M_E	5.98×10^{24} kg
Earth-Sun distance	1 AU	1.50×10^{11} m
Stirling's Approximation:	$\ln(N!) = N\ln(N) - N +$ (small corrections)	

1. Consider a very small hill on the Earth's surface that has the shape of a half sphere with radius R. A point on the hill is at the height $z = R\cos\theta$ (so that the top of the hill is at $\theta = 0$). A point mass m can slide on this hill without friction. The mass is initially at the top of the hill. At some moment, it receives a very small push in one direction, and it then starts sliding towards the bottom of the hill due to gravity. Compute the angle θ at which the mass m loses contact with the hill.

2. (i) Find the lowest relativistic correction to the classical kinetic energy of a particle of mass m moving with speed $v << c$.
(ii) Using the results of part (i), compute the leading order relativistic correction to the energy eigenvalues for a particle of mass m confined in the one dimensional infinite well potential:

$$V = \begin{cases} +\infty & , \quad x < 0 \\ 0 & , \quad 0 < x < L \\ +\infty & , \quad x > L \end{cases}$$

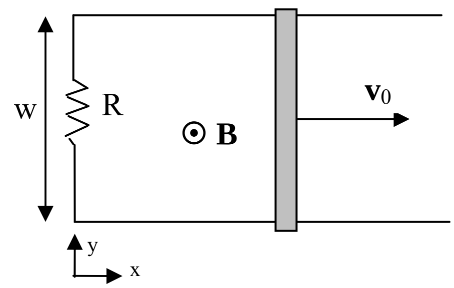
3. Cold interstellar molecular clouds often contain the molecule cyanogen (CN), whose first rotational excited states have an energy 4.7×10^{-4} eV compared to the ground state. There are three such excited states, all with the same energy. In 1941, studies of the absorption spectrum of starlight that passes through molecular clouds showed that for every 10 CN molecules that are in the ground state, three others are in the first excited state. To account for these data, astronomers suggested that the CN molecules might be in thermal equilibrium with some "reservoir" with a well-defined temperature.
   (i) Find the partition function Z.
   (ii) What is the temperature T of the reservoir in K?
   (iii) Given your result for T, what is the reservoir?

4. Consider a hydrogen atom located at $r = 0$. Assume that in addition to the proton Coulomb potential, the electron experiences a small short range spherically symmetric potential $U(r) = u a_B^3 \delta(\mathbf{r})$, where $\mathbf{r}$ is a 3D vector, a_B is the Bohr radius, and u is much smaller than the ionization energy of the hydrogen atom. Calculate the correction to the energy $\hbar\omega$ of the 1s-2p transition. (This can be a correction for the finite size of the proton in a hydrogen atom).

5. Consider the energy changes of a plane capacitor made of two parallel, horizontal metallic plates separated by air for two different cases:
   (i) In the first case, a capacitor with capacitance C_1 is charged by a battery to a voltage V and then disconnected. The distance between plates is slowly increased by an external force doing work $A > 0$, so that capacitance becomes $C_2 < C_1$. What is the change of the electrostatic energy U of capacitor during this process? Show that energy is conserved.
   (ii) In the second case, the battery remains connected to the capacitor as the distance between the plates is increased. What is the change in the electrostatic energy of the capacitor after the same work A is done to change the capacitance from C_1 to C_2? Show how energy is conserved in this case. What is different in this case from case (i)?

6. The International Space Station is in a circular orbit with an altitude of 500 km above the surface of the Earth.
   (i) Calculate the velocity and period of the station's orbit.
   (ii) Suppose a satellite is launched from the station with a relative velocity of one tenth the station's orbital velocity in the direction of the station's motion. Determine the semi-major axis of the new orbit, as well as the perigee (closest distance to the center of Earth), apogee (farthest distance from the center of Earth) and period of the new orbit. (Hint: the semi-major axis is the average of the perigee and apogee).

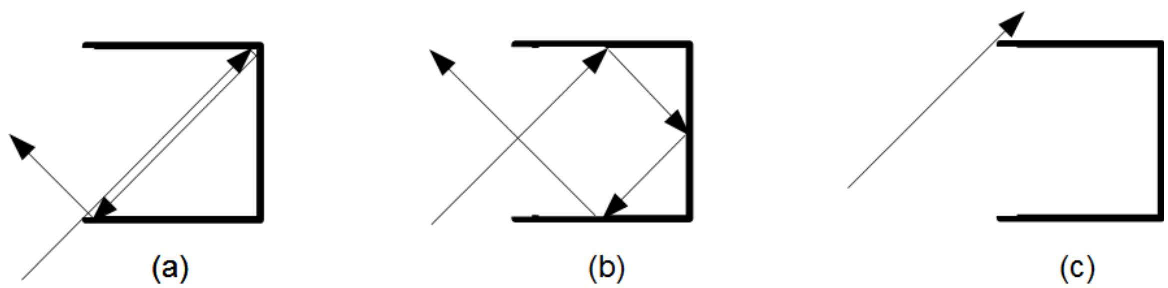
7. Consider a "sliding bar" generator made of a U-shaped wire with width w and a resistance R with the current closed by a movable bar of mass M and an initial velocity $\mathbf{v}_0$, as shown in the figure. A magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ points out of the page.
   (i) Find the emf generated and the current that flows through the circuit. Draw a diagram showing the direction of the current flow.
   (ii) Determine the magnetic force on the bar, and solve for its motion.
   (iii) Show that the energy dissipated by the resistor is equal to the kinetic energy lost by the bar.



8. In the lab frame, an incoming proton collides with a second proton at rest, giving rise to the process
$$p + p \rightarrow p + p + \pi^0$$
Such a process can occur only if the energy of the incident proton is above a given threshold energy. Compute the threshold energy of the incident proton in the center-of-mass (rest) frame, and in the lab frame. Recall that $m_pc^2 = 938$ MeV, and $m_{\pi^0}c^2 = 135$ MeV. (Hint: Consider using an appropriate relativistic invariant quantity to simplify the algebra).

9. In a plasma with equal concentrations of free positive and negative ions, screening modifies the Coulomb potential e^2/r of a point charge e to the Yukawa form, $V(r) = (e^2/r)\exp(-\kappa r)$. Here r is the distance from the point charge to the observation point and κ is the inverse screening radius.
   (i) Calculate the Fourier transform $V(q)$ of the Yukawa potential.
   (ii) What is Fourier transform $V_C(q)$ of the Coulomb potential?

10. Consider a square box of mass m whose interior surface is 100% reflective. Each edge of the box has a length L and the box lies on a frictionless surface. One side of the box is removed to allow light to enter the box. A beam from an ideal laser with a wavelength of λ and output power P is pointed at a 45^{o} angle from the horizontal towards the opposite corner of the box (figure a). The box gradually moves to the right causing the laser beam to sweep across the top of the box (figure b) until the laser no longer points into the box (figure c). If the box starts at rest, what will be the final momentum of the box? Ignore the small influence of the Doppler effect on the light.



University of Minnesota
School of Physics and Astronomy

GRADUATE WRITTEN EXAMINATION

Spring 2015 – PART II

Friday, January 16th, 2015 – 9:00 am to 1:00 pm

Part 2 of this exam consists of 5 problems of equal weight. You will be graded on all 5 problems.

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1. In semiconductors, electron states of donors are similar to the hydrogen atom because a donor has the same charge as a proton. The only difference is that the Coulomb potential of a donor $V(r) = -e^2/\kappa r$ contains the dielectric constant κ of the semiconductor and the effective electron mass m^*, which is usually much smaller than the free electron mass m. As a result, the ground state energy E and the effective Bohr radius a of a hydrogen-like donor state is modified from the ground state energy and Bohr radius of a hydrogen atom.

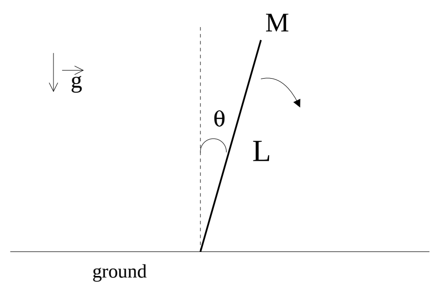
(i) Determine the ground state energy E for a hydrogen-like donor, and evaluate it for GaAs (κ = 12.5 and m^*= 0.07m).

(ii) Determine the effective Bohr radius a for a hydrogen-like donor and evaluate it for GaAs (κ = 12.5 and m^*= 0.07m). Write an expression for the ground state wave function.

(iii) Now consider a donor located at the surface $z = 0$ of the semiconductor, which occupies the half space $z > 0$, while for $z < 0$ there is vacuum. Assume that the electron cannot penetrate into the vacuum, so that its wave function vanishes at $z = 0$. On the other hand, for $z > 0$ the electron is subjected only to the Coulomb potential $V(r) = -e^2/\kappa r$ of the donor. Find the ionization energy of such a surface donor E_s using your knowledge of the wave functions of first excited states of the bulk donor hydrogen atom and compare it to that of the bulk donor energy E.

2. (i) Show that the moment of inertia of a (one dimensional) uniform rod of mass M and length L that rotates about one end is $I = ML^2/3$.

(ii) A uniform rod of length L and mass M is falling with one end fixed on the ground. Compute the classical differential equation governing the time evolution of $\theta(t)$ (see the figure). Linearize this equation in the limit $\theta \ll 1$.



(iii) Assume that, at the time $t = 0$, the rod has a very small initial angle θ_{in} and a very small initial (positive) angular speed $\dot{\theta}_{in}$. Estimate how long it takes the rod to fall by solving the linearized equation obtained in (ii) with these initial conditions. Use this equation to determine when $\theta = 1$. The time t_{fall} at which this happens is a good estimate for the time necessary for the rod to fall, starting with those initial conditions.

(iv) Classically, one can imagine the rod to be perfectly vertical, and at rest. Quantum mechanically, however, this is not possible. From the Lagrangian of the system, determine the conjugate variable of the angle θ, and use Heisenberg's uncertainty principle to obtain a lower bound on the product $\theta_{in} \times \dot{\theta}_{in}$. To do so, replace the uncertainty in the expectation value for one variable with the corresponding classical initial condition (namely, replace $\Delta\theta$ with θ_{in}, and analogously for $\dot{\theta}_{in}$).

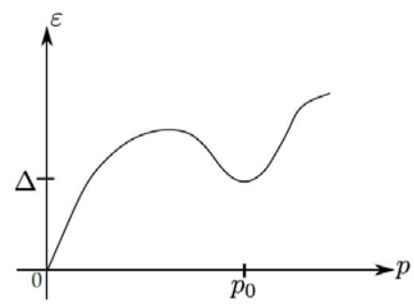
(v) Find the largest possible value for t_{fall} compatible with the lower bound that you have found in (iv).

(vi) Evaluate the result obtained in (v) for M = 50 g and L = 30 cm.

3. Consider an AC transmission line consisting of two thin parallel plates with width w (in the y direction) and separated by a distance d (in the x direction), with $d \ll w$ so that fringing fields can be neglected. One plate is grounded, and the other is at a potential given by $V = V_0 \cos(kz - \omega t)$. A current $I = I_0 \cos(kz - \omega t)$ flows in the $-z$ direction on the grounded plate and in the $+z$ direction on the other plate.
   (a) Determine the electric and magnetic fields in the region between the plates.
   (b) Using Faraday's Law, determine the relationship between the frequency ω and the wave number k, and find the ratio of the magnitudes of **E** and **B**.
   (c) Determine the Poynting flux between the plates, and integrate it over the cross section of the plates (i.e., the xy plane) to find the total energy flow. Express this in terms of V_0 and I_0.

4. Consider a system of two coupled pendula, each with mass m and length a, hanging from points a distance d apart. The pendula are coupled by a spring with spring constant k and an equilibrium length that is equal to the separation of the pendula. (That is, the spring is relaxed when the pendula are each hanging straight down.). Consider only motions in the plane, and neglect the masses of the rods and the spring. Find the frequencies of small oscillation of this system, and determine the normal modes. (Hint: you only need to consider the horizontal displacements when considering the potential energy of the spring.)

5. To explain experimental data on the low temperature specific heat of super fluid helium, Landau (1938) conjectured that the low-energy spectrum of its Bose excitations, $\varepsilon(p)$, has a peculiar form, which goes as $\varepsilon(p) = sp$ for $p \ll p_0$, reaches a maximum at some $p < p_0$ and then goes through a minimum at $p = p_0$. Near the minimum it can be described by $\varepsilon(p) = \Delta + (p - p_0)^2 / 2\sigma$. Excitations in the first (linear) part of the spectrum are called phonons, excitations in the second (parabolic) part are called rotons. Later it became known that the sound velocity, $s = 239$ m/s, $\Delta/k_B = 8.65$ K, $p_0 = 1.92\hbar/\text{Å}$, where Å is one Ångström, and $\sigma = 0.16 m_{He}$, where m_{He} is the helium atom mass.



   (i) Calculate the phonon contributions to the specific heat per unit volume. Use

$$I = \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

   (ii) Calculate the roton contributions to the specific heat per unit volume, assuming that $k_B T << \Delta$ and $k_B T \sigma \ll p_0^2$, so that only a thin spherical shell with radius p_0 dominates the integration in momentum space.
   (iii) Estimate the temperature where crossover between the phonon and roton contributions happens for the parameters given above. One can proceed analytically by equating both contributions, while assuming that $k_B T = \Delta$ everywhere except in the exponential factor. One can also check that $\sigma\Delta \ll p_0^2$, (It was observation of this crossover that led Landau to his conjecture.)