Directional Analysis of Seismic Data

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1 Introduction

We develop a formalism for estimating the modal content of a seismic wave field, using a limited number of seismic measurements.

2 Body Waves

2.1 Formalism

Let us start with the shear and pressure wave fields, $\vec{s}(\vec{x},t)$ and $\vec{p}(\vec{x},t)$ (others can be added as well). The plane wave expansions of these are:

$$\vec{s}(\vec{x},t) = \sum_A \int df d\hat{\Omega}\; S_A(f,\hat{\Omega})\; \vec{e}_A(\hat{\Omega})\; e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/v_s)} \tag{1}$$

$$\vec{p}(\vec{x},t) = \int df d\hat{\Omega}\; P(f,\hat{\Omega})\; \hat{\Omega}\; e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/v_p)} \tag{2}$$

Here, $\hat{\Omega}$ denotes the wave propagation direction, f is frequency, $S_A(f,\hat{\Omega})$ is the amplitude of the shear wave of polarization A, defined by the unit vector $\vec{e}_A(\hat{\Omega})$ that is perpendicular to the wave propagation direction $\hat{\Omega}$), $P(f,\hat{\Omega})$ is the amplitude of the pressure wave which has a single polarization in the longitudinal direction ($\hat{\Omega}$), and v_s and v_p are the speeds of the shear and pressure waves respectively. We can then define two-point correlations:

$$\langle S_A^*(f,\hat{\Omega})\; S_{A'}(f',\hat{\Omega}')\rangle = \delta_{AA'}\; \delta(f-f')\; \delta^2(\hat{\Omega},\hat{\Omega}')\; H_{S,A}(f,\hat{\Omega}) \tag{3}$$

$$\langle P^*(f,\hat{\Omega})\; P(f',\hat{\Omega}')\rangle = \delta(f-f')\; \delta^2(\hat{\Omega},\hat{\Omega}')\; H_P(f,\hat{\Omega}) \tag{4}$$

$$\langle S_A^*(f,\hat{\Omega})\; P(f',\hat{\Omega}')\rangle = 0 \tag{5}$$

Here, δ's denote the Kronecker or Dirac delta functions, and H's denote the power in the shear waves of polarization A or in pressure waves. These assumed two-point correlations essentially state that waves at different frequencies, from different directions, and of different polarization (shear or pressure) are all uncorrelated.

Seismometer i at a location $\vec{x}$ then measures in channel $\hat{\alpha}$:

$$d_{i,\alpha}(\vec{x},t) = (\vec{s}(\vec{x},t) + \vec{p}(\vec{x},t)) \cdot \hat{\alpha} \tag{6}$$

where α could be x,y, or z. In principle we should add the seismometer noise to this measurement, but we will assume that the seismic noise floor is significantly higher than the instrument noise. Now we can compute the cross-correlation between channels of two seismometers located at different locations (we denote the channels by α and β, they take values x, y, z and in general need not be the same):

$$\begin{aligned}
\langle Y_{i\alpha,j\beta} \rangle &= \int_{-T/2}^{T/2} dt\ d_{i\alpha}(\vec{x}_i,t) d_{j\beta}(\vec{x}_j,t) &(7)\\
&= \int_{-T/2}^{T/2} dt \int df d\hat{\Omega} \left(\sum_A H_{S,A}(f,\hat{\Omega}) e_{A,\alpha}(\hat{\Omega}) e_{A,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \right. \\
&+ \left. H_P(f,\hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_p} \right) &(8)
\end{aligned}$$

where we have defined projections $e_{A,\alpha}(\hat{\Omega}) = \vec{e}_A(\hat{\Omega}) \cdot \hat{\alpha}$ and $\Omega_\alpha = \hat{\Omega} \cdot \hat{\alpha}$, and $\Delta\vec{x} = \vec{x}_i - \vec{x}_j$. The time integral is trivial. Also, we can perform the analysis in a small frequency bin Δf so that the frequency integral is also simple (we add a factor of 2 when switching to integration over frequencies between 0 and $+\infty$):

$$\begin{aligned}
\langle Y_{i\alpha,j\beta} \rangle &= 2T\Delta f \int d\hat{\Omega} \left(\sum_A H_{S,A}(\hat{\Omega}) e_{A,\alpha}(\hat{\Omega}) e_{A,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \right. \\
&+ \left. H_P(\hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_p} \right) &(9)
\end{aligned}$$

where we have suppressed the frequency dependence of the shear and pressure wave amplitudes. We can then parameterize the amplitudes in terms of whatever basis $\{Q_a(\hat{\Omega})\}$ would be most useful:

$$H_{S,A}(\hat{\Omega}) = \sum_a S_{A,a}\, Q_a(\hat{\Omega}) \tag{10}$$

$$H_P(\hat{\Omega}) = \sum_a P_a\, Q_a(\hat{\Omega}) \tag{11}$$

One useful basis may be the spherical harmonics, $Q_{lm}(\hat{\Omega}) = Y_{lm}(\hat{\Omega})$. Another basis choice could be "pixels", i.e. specific propagation directions $Q_{\hat{\Omega}_0}(\hat{\Omega}) = \delta(\hat{\Omega} - \hat{\Omega}_0)$. Either way, we can define

$$\begin{aligned}
\gamma_{S1a} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) e_{1,\alpha}(\hat{\Omega}) e_{1,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \\
\gamma_{S2a} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) e_{2,\alpha}(\hat{\Omega}) e_{2,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \\
\gamma_{Pa} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_p}
\end{aligned} \tag{12}$$

Note we suppressed the indices $ij\alpha\beta$. We can then write

$$\begin{aligned}
\langle Y_{i\alpha,j\beta} \rangle &= 2T\Delta f (S_{1a}\gamma_{S1a} + S_{2b}\gamma_{S2b} + P_c \gamma_{Pc}) \\
&= S_d \gamma_d
\end{aligned} \tag{13}$$

where the repeated indices are summer over (and the sum over the index d in the last line includes all 3 sums in the previous line). The goal of the analysis is to estimate the coefficients S_{1a}, S_{2b}, P_c (or, equivalently, S_d). We define the likelihood as

$$\mathcal{L} \propto \exp\Big(-(Y_i^* - \gamma_{id}^* S_d) N^{-1} (Y_i - \gamma_{id} S_d) \Big) \tag{14}$$

where i now runs over all detector/channel pairs, and d runs over all basis elements. The covariance matrix N in our case is simple - we can assume that all detectors and channels have similar noise floors, constant in time, so N then becomes proportional to the identity matrix and we can ignore it in likelihood maximization. The best estimate is then given by

$$\vec{S} = (\gamma^{T*}\gamma)^{-1} \gamma^* \vec{Y} \tag{15}$$

where in the last line we think of γ as a matrix of elements γ_{id}. So the problem reduces to computing the γ matrix, which can be done once the basis is chosen using Eq. 12. Note that it is straightforward to add other components of the seismic wave field.

3 Surface Waves

3.1 Love Waves

Following the standard theory in seismology (Aki and Richards, 2009), Love waves–horizontally-polarized shear waves often noted as SH-waves–do not exist in a homogeneous halfspace, so let us consider these waves in an isotropic and vertically heterogeneous halfspace with corresponding elastic moduli that are smoothly-varying with depth. Then, assuming a plane-wave solution as above and appropriate boundary conditions (namely $l_1 \to 0$ for suffiently large z and $\mu(z)\frac{dl_1}{dz} = 0$ at the free surface), the equation of motion takes the form:

$$-\omega^2 \rho(z) l_1 = \frac{d}{dz}\left[\mu(z)\frac{dl_1}{dz}\right] - k^2 \mu(z) l_1, \tag{16}$$

where l_1, most-generally, is the eigenfunction that captures the displacement field's dependence on depth, frequency, and wavenumber; i.e. $l_1 = l_1(z, \omega, k)$. Equation 16 has dependence on the shear modulus, $\mu(z)$, and density, $\rho(z)$, of the medium; this must be accounted for by using an appropriate shear-wave velocity profile. By definition, the shear-wave velocity is:

$$\beta(z) = \sqrt{\frac{\mu(z)}{\rho(z)}}. \tag{17}$$

Assuming density is constant with depth, i.e. $\rho(z) = \rho_0$, we consider those profiles modelled by a power-law: $\beta(z) \sim z^{-\alpha}$. Furthermore, Haney and Tsai (2015) suggests an approximate eigenfunction solution for fundamental-mode Love waves: $l_1 \sim e^{-akz}$. In keeping consistent with the rest of the note, the eigenfunction in Haney and Tsai (2015) would be: $l_1 = l_1(z, \omega, k) \sim l_1(z, f, v_l) = e^{-2\pi a \frac{fz}{v_l}}$ if $k = \omega/v_l$ and $\omega = 2\pi/f$. Note its dependence on Love wave velocity, v_l; this value is usually obtained experimentally. Because a depends on α, a scan through different values of shear-wave power-law index, $\alpha \in \begin{bmatrix} 0.250 & 0.275 & 0.300 & 0.325 & 0.350 & 0.375 & 0.400 \end{bmatrix}$, yields $a = 0.85 \pm 0.09$.

3.1.1 Formalism

Consider a single Love wave propagating in the $\hat{\Omega}$ direction with particle displacement perpendicular and horizontally polarized with repsect to the line of propagation. We begin with the displacement field modified from Lay and Wallace (1995), Aki and Richards (2009):

$$\vec{l}(\vec{x},t) = l_1 \ \cos(\vec{k}\cdot\vec{x} - \omega t) \ \vec{e}_H(\hat{\Omega}), \tag{18}$$

where the naming conventions follow that of shear waves presented earlier, i.e. here $A = H$ to signify the horizontal polarization of the wave. For the sake of argument, we assume the fundamental-mode of Love waves contributes more significantly than other modes, which leads to a frequency-dependent eigenfunction solution: $l_1(z, f, v_l) = e^{-2\pi a \frac{fz}{v_l}}$ as in 3.1. This is done to illustrate the attenuation of these waves as one goes further underground. Furthermore, the plane-wave expansion of this displacement field is:

$$\vec{l}(\vec{x},t) = \int df \ d\hat{\Omega} \ e^{-2\pi a \frac{fz}{v_l}} \ L(f,\hat{\Omega}) \ \vec{e}_H(\hat{\Omega}) \ e^{2\pi i f\left(t - \frac{\hat{\Omega}\cdot\vec{x}}{v_l}\right)}. \tag{19}$$

Then, the two-point correlation is:

$$\langle L^*(f,\hat{\Omega}) \ L^{'}(f',\hat{\Omega}')\rangle = \delta(f-f') \ \delta^2(\hat{\Omega},\hat{\Omega}') \ H_L(f,\hat{\Omega}), \tag{20}$$

and the measurement in channel α of seismometer i would be:

$$d_{i,\alpha}(\vec{x},t) = \vec{l}(\vec{x},t)\cdot\hat{\alpha}, \tag{21}$$

where α can be (x, y, or z), allows us to compute the cross-correlation Y (between channel α of seismometer i and channel β of seismometer j):

$$\langle Y_{i\alpha,j\beta}\rangle = 2T\Delta f \int d\hat{\Omega} \ e^{-2\pi a f(z_i+z_j)/v_l} H_L(\hat{\Omega}) \ e_{H,\alpha}(\hat{\Omega}) \ e_{H,\beta}(\hat{\Omega}) \ e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{x}/v_l}, \tag{22}$$

where the frequency dependence in Eq. 19 was ignored as in the preceding analysis and $e_{H,\alpha}(\hat{\Omega}) = \vec{e}_H(\hat{\Omega})\cdot\hat{\alpha}$. Expanding the spatial function in some basis, e.g. pixels or spherical harmonics:

$$H_L(\hat{\Omega}) = \sum_a L_a \ Q_a(\hat{\Omega}), \tag{23}$$

allows us to write the γ-functions:

$$\gamma_{La} = \int d\hat{\Omega} \ Q_a(\hat{\Omega}) \ e_{H,\alpha}(\hat{\Omega}) \ e_{H,\beta}(\hat{\Omega}) \ e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{x}/v_l}, \tag{24}$$

which, finally, enables us to write the cross-correlation as a sum over these new γ-functions:

$$\langle Y_{i\alpha,j\beta}\rangle = 2T\Delta f \; e^{-2\pi a f(z_i+z_j)/v_l} \; L_a \; \gamma_{La}. \tag{25}$$

3.2 Rayleigh waves

Now, let's try to extend this formalism to Rayleigh waves, characterized by the displacement field

$$r(\vec{x},t) = r_1(\omega,k,z)\cos(\omega t - \vec{k}\cdot\vec{x})\hat{k} + r_2(\omega,k,z)\sin(\omega t - \vec{k}\cdot\vec{x})\hat{z}. \tag{26}$$

This is meant to capture the retrograde motion of a particle in a plane defined by the wave propagation direction and the vertical axis. In general, we expect the amplitude to exponentially decay, although the decay rate and amount is frequency- and material-dependent. This behavior is encapsulated by the eigenfunction coefficients: r_1 (horizontal) and r_2 (vertical). The plane wave expansion in the above formalism is then

$$\vec{r}(\vec{x},t) = \int df d\hat{\Omega}\, R(f,\hat{\Omega})\, e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/v_r)} \left(r_1\hat{\Omega} + r_2 e^{i\pi/2}\hat{z}\right). \tag{27}$$

Here, $R(f,\hat{\Omega})$ is the amplitude of the Rayleigh wave coming from direction $\hat{\Omega}$ with frequency f, and v_r is the Rayleigh wave velocity.

We proceed to define a two-point correlation, in simplest form to start with:

$$\langle R^*(f,\hat{\Omega})\, R(f',\hat{\Omega}')\rangle = \delta(f-f')\,\delta^2(\hat{\Omega},\hat{\Omega}')\, H_R(f,\hat{\Omega}). \tag{28}$$

Similarly to the case of body waves, the measurement of seismometer i (at location $\vec{x}_i$ and time t) in channel α is given by

$$d_{i,\alpha}(\vec{x}_i,t) = \vec{r}(\vec{x}_i,t)\cdot\hat{\alpha}. \tag{29}$$

Using this, we compute the cross correlation estimator Y between detector i, channel α and detector j, channel β, ignoring the frequency dependence:

$$\langle Y_{i\alpha j\beta}\rangle = 2T\Delta f \int d\hat{\Omega}\, H_R(\hat{\Omega}) e^{2\pi i f\hat{\Omega}\cdot\Delta\vec{x}/v_r} \times \tag{30}$$

$$\left[(r_1\hat{\Omega}\cdot\hat{\alpha} + r_2 e^{i\pi/2}\hat{z}\cdot\hat{\alpha})(r_1\hat{\Omega}\cdot\hat{\beta} + r_2 e^{-i\pi/2}\hat{z}\cdot\hat{\beta})\right] \tag{31}$$

Next, we expand the spatial function in some basis Q_d (pixel or spherical harmonics):

$$H_R(\hat{\Omega}) = \sum_d R_d \, Q_d(\hat{\Omega}). \tag{32}$$

Here, H_R represents the power in Rayleigh waves coming from direction $\hat{\Omega}$. Using this decomposition, we calculate the gamma functions:

$$\gamma^R_{d,i\alpha j\beta} = \int d\hat{\Omega} \, Q_d(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_r} \times \tag{33}$$

$$\left[(r_1 \hat{\Omega}\cdot\hat{\alpha} + r_2 e^{i\pi/2} \hat{z}\cdot\hat{\alpha})(r_1 \hat{\Omega}\cdot\hat{\beta} + r_2 e^{-i\pi/2} \hat{z}\cdot\hat{\beta}) \right]. \tag{34}$$

Finally, we write the estimator as a sum over the gamma functions:

$$\langle Y_{i\alpha j\beta} \rangle = 2T\Delta f \, R_d \gamma^R_{d,i\alpha j\beta} \tag{35}$$

3.2.1 Simple model

Here, we formulate a simple model of the Rayleigh wave eigenfunctions:

$$r_1(\omega, k, z) = e^{-z/\alpha} \tag{36}$$

$$r_2(\omega, k, z) = \epsilon e^{-z/\alpha} \tag{37}$$

In this model, both the horizontal and vertical amplitudes decay with depth; this behavior is controlled by a constant α, which we expect to be proportional to the wavelength of the Rayleigh wave. The ϵ parameter controls the relative vertical and horizontal amplitudes.

Using this formulation results in the following gamma functions:

$$\gamma^R_{d,i\alpha j\beta} = \int d\hat{\Omega} \, Q_d(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_r} e^{-(z_i+z_j)/\alpha} \times \tag{38}$$

$$\left[(\hat{\Omega}\cdot\hat{\alpha} + \epsilon e^{i\pi/2} \hat{z}\cdot\hat{\alpha})(\hat{\Omega}\cdot\hat{\beta} + \epsilon e^{-i\pi/2} \hat{z}\cdot\hat{\beta}) \right], \tag{39}$$

which may be inserted into the above equations in order to explicitly calculate the cross-correlation.

3.2.2 Biexponential model

For the case of a half-space where the material properties are only functions of depth, the equations of motion for Rayleigh waves may be solved analytically. This results in a biexponential functional form for the eigenfunctions (see Haney and Tsai (2015)):

$$r_1(\omega, k, z) = C_1 e^{-a_1 k z} + C_2 e^{-a_2 k z} \tag{40}$$

$$r_2(\omega, k, z) = C_3 e^{-a_3 k z} + C_4 e^{-a_4 k z} \tag{41}$$

We generally take C_1 to be 1 in order to break some degeneracy between the coefficients. We can also write k in terms of the phase velocity (c_p), since these eigenfunctions describe a particular frequency of Rayleigh wave.

$$r_1 = \quad e^{-a_1 \omega z / c_p} + C_2 e^{-a_2 \omega z / c_p} \tag{42}$$

$$r_2 = C_3 e^{-a_3 \omega z / c_p} + C_4 e^{-a_4 \omega z / c_p} \tag{43}$$

We note that there is still some degeneracy under interchange of C_3 and a_3 with C_4 and a_4, which can be eliminated by fixing the ranges of these parameters.

These eigenfunctions may be inserted into Eq. 33 in order to calculate the gamma functions. We also note that that we use the phase velocity c_p in the gamma functions for this model (in place of a general Rayleigh wave velocity v_r). This functional form allows for the vertical and horizontal amplitudes to vary independently and does not restrict them to be monotonically decreasing with depth; it has been studied in detail in Prestegard (2016), including estimates for the eigenfunction parameters.

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