Love Wave Formalism

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0.0.1 Theory and Eigenfunction Approximation

Following the standard theory in seismology (Aki and Richards, 2009), Love waves–horizontally-polarized shear waves often noted as SH-waves–do not exist in a homogeneous halfspace, so let us consider these waves in an isotropic and vertically heterogeneous halfspace with corresponding elastic moduli that are smoothly-varying with depth. Then, assuming a plane-wave solution as above and appropriate boundary conditions (namely $l_1 \to 0$ for suffiently large z and $\mu(z)\frac{dl_1}{dz} = 0$ at the free surface), the equation of motion takes the form:

$$-\omega^2 \rho(z) l_1 = \frac{d}{dz}\left[\mu(z)\frac{dl_1}{dz}\right] - k^2 \mu(z) l_1, \tag{1}$$

where l_1, most-generally, is the eigenfunction that captures the displacement field's dependence on depth, frequency, and wavenumber; i.e. $l_1 = l_1(z, \omega, k)$. Equation 1 has dependence on the shear modulus, $\mu(z)$, and density, $\rho(z)$, of the medium; this must be accounted for by using an appropriate shear-wave velocity profile. By definition, the shear-wave velocity is:

$$\beta(z) = \sqrt{\frac{\mu(z)}{\rho(z)}}. \tag{2}$$

Assuming density is constant with depth, i.e. $\rho(z) = \rho_0$, we consider those profiles modelled by a power-law: $\beta(z) \sim z^{-\alpha}$. Furthermore, Haney and Tsai (2015) suggests an approximate eigenfunction solution for fundamental-mode Love waves: $l_1 \sim e^{-akz}$. In keeping consistent with the rest of the note, the eigenfunction in Haney and Tsai (2015) would be: $l_1 = l_1(z, \omega, k) \sim l_1(z, f, v_l) = e^{-2\pi a \frac{fz}{v_l}}$ if $k = \omega / v_l$ and $\omega = 2\pi / f$. Note its dependence on Love wave velocity, v_l; this value is usually obtained experimentally.

Because a depends on α, a scan through different values of shear-wave power-law index, $\alpha \in \begin{bmatrix}0.250 & 0.275 & 0.300 & 0.325 & 0.350 & 0.375 & 0.400\end{bmatrix}$, yields $a = 0.85 \pm 0.09$.

0.0.2 Recovery Method

Consider a single Love wave propagating in the $\hat{\Omega}$ direction with particle displacement perpendicular and horizontally polarized with repsect to the line of propagation. We begin with the displacement field modified from Lay and Wallace (1995), Aki and Richards (2009):

$$\vec{l}(\vec{x}, t) = l_1 \ \cos(\vec{k} \cdot \vec{x} - \omega t) \ \vec{e}_H(\hat{\Omega}), \tag{3}$$

where the naming conventions follow that of shear waves presented earlier, i.e. here $A = H$ to signify the horizontal polarization of the wave. For the sake of argument, we assume the fundamental-mode of Love waves contributes more significantly than other modes, which leads to a frequency-dependent eigenfunction solution: $l_1(z, f, v_l) = e^{-2\pi a \frac{fz}{v_l}}$ as in 0.0.1. This is done to illustrate the attenuation of these waves as one goes further underground. Furthermore, the plane-wave expansion of this displacement field is:

$$\vec{l}(\vec{x}, t) = \int df \ d\hat{\Omega} \ e^{-2\pi a \frac{fz}{v_l}} \ L(f, \hat{\Omega}) \ \vec{e}_H(\hat{\Omega}) \ e^{2\pi i f \left(t - \frac{\hat{\Omega} \cdot \vec{x}}{v_l}\right)}. \tag{4}$$

Then, the two-point correlation is:

$$\langle L^*(f, \hat{\Omega}) \ L^{'}(f', \hat{\Omega}') \rangle = \delta(f - f') \ \delta^2(\hat{\Omega}, \hat{\Omega}') \ H_L(f, \hat{\Omega}), \tag{5}$$

and the measurement in channel α of seismometer i would be:

$$d_{i,\alpha}(\vec{x}, t) = \vec{l}(\vec{x}, t) \cdot \hat{\alpha}, \tag{6}$$

where α can be (x, y, or z), allows us to compute the cross-correlation Y (between channel α of seismometer i and channel β of seismometer j):

$$\langle Y_{i\alpha,j\beta} \rangle = 2T\Delta f \int d\hat{\Omega} \ e^{-2\pi a f(z_i+z_j)/v_l} H_L(\hat{\Omega}) \ e_{H,\alpha}(\hat{\Omega}) \ e_{H,\beta}(\hat{\Omega}) \ e^{2\pi i f \hat{\Omega} \cdot \Delta\vec{x}/v_l}, \tag{7}$$

where the frequency dependence in Eq. 4 was ignored as in the preceding analysis and $e_{H,\alpha}(\hat{\Omega}) = \vec{e}_H(\hat{\Omega}) \cdot \hat{\alpha}$. Expanding the spatial function in some basis, e.g. pixels or spherical harmonics:

$$H_L(\hat{\Omega}) = \sum_a L_a \ Q_a(\hat{\Omega}), \tag{8}$$

allows us to write the γ-functions:

$$\gamma_{La} = \int d\hat{\Omega}\ Q_a(\hat{\Omega})\ e_{H,\alpha}(\hat{\Omega})\ e_{H,\beta}(\hat{\Omega})\ e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_l}, \tag{9}$$

which, finally, enables us to write the cross-correlation as a sum over these new γ-functions:

$$\langle Y_{i\alpha,j\beta}\rangle = 2T\Delta f\ e^{-2\pi a f(z_i+z_j)/v_l}\ L_a\ \gamma_{La}. \tag{10}$$

References

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Matthew M. Haney and Victor C. Tsai. Nonperturbational surface-wave inversion: A Dix-type relation for surface waves. *Geophysics*, 80(6):EN167–EN177, 2015. URL doi:10.1190/geo2014-0612.1.

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