Terrestrial gravitational noise on a gravitational wave antenna

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A random gravitational force can be generated by seismic noise, by atmospheric acoustic noise, and by moving massive bodies. An estimate of the gravitational power spectrum at a point on the Earth is given. Such a force is an important source of noise in an interferometric gravitational wave antenna below $f = 10$ Hz.

INTRODUCTION

A new generation of interferometric antennas for the detection of gravitational radiation of cosmic sources is being planned. Detection of gravitational waves is performed by measuring the relative displacements of several nearly free masses which carry the mirrors defining a Michelson interferometer. The measured quantity is the difference (as a function of time) in the lengths of the two orthogonal arms of the interferometer. In principle, this form of antenna can be sensitive down to quite low frequencies. In practice, various noise sources will limit the useful bandpass. One form of noise is random gravitational forces. This is a particularly important form of noise, since gravitational forces cannot be shielded, even in principle.

The effect of random gravitational forces, or gravity gradient noise, has been an important consideration in all sensitive gravitational experiments. (See the review by Everitt.[1]) Suzuki and Hirakawa[2] pointed out the importance of nonradiative fluctuations in the local gravitational field for low-frequency gravitational-wave experiments. In the present work we make a quantitative estimate of this noise source, with special emphasis on the limitations which it places on the performance of interferometric gravitational-wave antennas with baselines of 1 km or greater.

Sources of random gravitational forces can be grouped into two categories. One sort is fluctuations in the density of a medium (air or earth) surrounding the antenna. The other kind is the motion of isolated massive bodies in the vicinity of the antenna. (These are not completely distinct categories—an airplane generates sound, so both sorts of sources are present.)

FORCES DUE TO DENSITY FLUCTUATIONS IN A MEDIUM

We can make an estimate of the magnitude of the forces due to density fluctuations by using dimensional analysis. First, note that gravitational forces produce on a test body a force that is proportional to the mass of the test body, or in other words, a force per unit mass (or acceleration) that is independent of the mass of the test body. We will want to solve the problem in the frequency domain, so we write the acceleration as $\omega^2 x$. By Newton's law of gravity, the gravitational acceleration must be proportional to the constant of gravitation G, to the density of the medium ρ, and to some measure of the fluctuations. If we are considering forces from seismic motion, the data we have at our disposal are spectra of the displacement of the Earth, X. Since the quantity $G\rho X$ has the dimensions of force per unit mass, then Newton's second law gives $\omega^2 x = AG\rho X$, where A is a dimensionless constant which depends on the geometry of the problem. For the case of air-pressure fluctuations, the measurements give the fractional pressure fluctuation $\Delta p/p$. If the characteristic length of a coherent pressure fluctuation is λ, then Newton's second law gives in this case $\omega^2 x = BG\rho\lambda\Delta p/p$, where B is another dimensionless constant. In the remainder of this section, we will make a simple model to estimate the value of these dimensionless constants. Finally, we will plug in measured values of the disturbance spectra to determine the approximate magnitude of the random gravitational noise in an interferometric antenna.

The air or the Earth fills a half-space around the antenna. (Refer to Fig. 1). Imagine that in one region there is a fluctuation $\Delta M(t) = M(t) - \langle M(t)\rangle$. This causes a fluctuating force on a test mass m equal to

$$\frac{\vec{F}(t)}{m} = \frac{G\Delta M(t)}{r^2}\hat{r} . \tag{1}$$

Taking the x component and transforming the equation from the time domain to the frequency domain we find

$$\frac{F_x}{m} = G\Delta M(\omega)\frac{\cos\theta}{r^2} . \tag{2}$$

We can substitute the equation of motion of the test mass (assuming it to be suspended from some sort of spring and damper giving it resonant frequency ω_0 and damping time

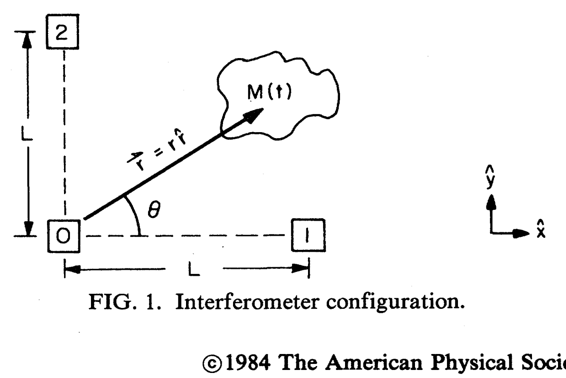

FIG. 1. Interferometer configuration.

τ) and finally take the squared modulus of both sides to yield

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|x(\omega)|^2=G^2|\Delta M(\omega)|^2\frac{\cos^2\theta}{r^4}. \quad (3)$$

Now consider that the half-space is filled with equivalent regions of fluctuating mass. Assume that the size of a coherently fluctuating region is of order $\lambda/2$, where $\lambda=v_s/f$ is the acoustic wavelength. Assume also that the fluctuations in different regions are independent of each other. (The physical validity of these simplifying assumptions will be discussed below.) Then the random forces from the different regions add in quadrature. So we have

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|x(\omega)|^2 =G^2|\Delta M(\omega)|^2\sum\frac{\cos^2\theta}{r^4}, \quad (4)$$

where the sum is taken over all the regions in the half-space. We evaluate the sum by approximating it as an integral. It converges if we introduce an inner cutoff radius $r_{\min}=\lambda/4$. We find

$$\sum\frac{\cos^2\theta}{r^4}=\frac{64\pi}{3\lambda^4}=\frac{64}{3v_s^4}\left[\frac{\omega}{2\pi}\right]^4. \quad (5)$$

(The fact that we need to introduce an inner cutoff on the integral indicates that the random gravitational force is dominated strongly by the nearest few coherently fluctuating regions. The choice $r_{\min}=\lambda/4$ is the natural one in the context of our model. Choosing a number much smaller would violate the assumption that the sum over regions can be approximated by an integral. Thus, although this step cannot be regarded as exact, it is unlikely to give an answer which underestimates the effect by a large factor.) Upon substitution into the previous equation we obtain

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|x(\omega)|^2=\frac{4G^2}{3\pi^3v_s^4}\omega^4|\Delta M(\omega)|^2. \quad (6)$$

The interferometer can only measure differences in the separations of two pairs of test masses. Consider the case $\lambda\ll L$, where L is the nominal separation between two masses. Then, since the random force is dominated by the regions within a few wavelengths of the test mass, the forces on any two test masses are uncorrelated, so they add in quadrature. Similarly, the two orthogonal arms of the interferometer are uncorrelated, so altogether the difference in separations is four times (in power) the motion of an individual mass, or

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2=\frac{16G^2}{3\pi^3v_s^4}\omega^4|\Delta M(\omega)|^2. \quad (7)$$

The baseline lengths being considered for the next generation of interferometric antennas are in the range 1—10 km. The highest sound speed present is that of P waves in the Earth, of order 8 km/sec. The short-wavelength approximation should be valid for frequencies above 10 Hz. We treat the long-wavelength approximation in the Appendix.

We still need to cast these formulas in terms of observables. We can write, for air-pressure fluctuations,

$$|\Delta M(\omega)|^2=V^2|\Delta\rho(\omega)|^2=\tfrac{1}{2}\left[\frac{\lambda}{2}\right]^6\frac{\rho_a^2|\Delta p(\omega)|^2}{p_a^2}, \quad (8)$$

where the extra factor of $\frac{1}{2}$ approximates $1/\gamma^2$ for adiabatic compression of air. Finally,

$$|\Delta M(\omega)|^2=\tfrac{1}{2}(\pi v_s)^6\frac{\rho_a^2}{p_a^2}\frac{|\Delta p(\omega)|^2}{\omega^6}. \quad (9)$$

Thus, the path-length-difference fluctuations due to air-pressure fluctuations are

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2 =\frac{8\pi^3}{3}G^2v_s^2\frac{\rho_a^2}{p_a^2}\frac{|\Delta p(\omega)|^2}{\omega^2}. \quad (10)$$

The case of P waves in the Earth is analogous, except that it is a displacement rather than a pressure variation that is usually measured. We find

$$|\Delta M(\omega)|^2=\frac{\pi}{16}\rho_e^2\lambda^4|\Delta X(\omega)|^2=\pi^5v^4\rho_e^2\frac{|\Delta X(\omega)|^2}{\omega^4}, \quad (11)$$

where ΔX is the displacement of a point in the Earth from its equilibrium position. Thus we have for the path-length difference

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2 =\frac{16\pi^2}{3}G^2\rho_e^2|\Delta X(\omega)|^2. \quad (12)$$

In addition, vertical motion at the earth-air interface causes random gravitational forces through the replacement of a volume of low-density air with high-density earth. Consistent with our previous model, we here make the assumption that the area of a coherent fluctuation on the Earth's surface should be of order $(\lambda/2)^2$. The sum over fluctuating regions must here be carried out over a plane instead of a half-space. We find

$$\sum\frac{\cos^2\theta}{r^4}=\frac{8\pi}{\lambda^4}=\frac{\omega^4}{2\pi^3v_s^4}. \quad (13)$$

So in this case

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2=\frac{2G^2}{\pi^3v_s^4}\omega^4|\Delta M(\omega)|^2. \quad (14)$$

Now

$$|\Delta M(\omega)|^2=\rho_e^2\left[\frac{\lambda}{2}\right]^4|\Delta Z(\omega)|^2=\pi^4 v_s^4\rho_e^2\frac{|\Delta Z(\omega)|^2}{\omega^4}\,, \quad (15)$$

where ΔZ is the vertical motion of the Earth's surface. The path-length-difference due to this mechanism is

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2=2\pi G^2\rho_e^2|\Delta Z(\omega)|^2\,. \quad (16)$$

This is roughly an order of magnitude smaller than the effect due to density fluctuations distributed through the Earth.

One might wonder whether our unphysical assumption that the fluctuations in neighboring regions are uncorrelated leads us to underestimate the random gravitational forces. In fact, the difference between the two models is not very large. An example will suffice to illustrate the point. For a half-plane wave in air with a root-mean-square amplitude $\Delta\rho(\omega)$, it is easy to write down (by analogy with the electric field inside a parallel-plate capacitor) that the force per unit mass is

$$\ddot{x}(\omega)=2\sqrt{2}G\lambda\Delta\rho(\omega)=\frac{4\sqrt{2}\pi v}{\omega}G\Delta\rho(\omega)\,. \quad (17)$$

The path-length difference $\Delta\ddot{x}(\omega)$ may be up to four times this size, or may cancel, depending on the relative phases of the wave at the three test masses. This is to be compared to

$$\Delta\ddot{x}(\omega)=\sqrt{4/3}G\lambda\Delta\rho(\omega)=\frac{4\pi}{3}\frac{v}{\omega}G\Delta\rho(\omega)\,, \quad (18)$$

which is the analogous quantity computed under the assumption of uncorrelated regions. The difference is small.

One can imagine another example where correlation lengths much longer than a wavelength threaten to drastically increase the amplitude of random gravitational forces. If we consider a P wave arriving at the antenna from a direction near the nadir, then it appears that vertical motions of the Earth's surface might be generated which are coherent over areas much larger than $(\lambda/2)^2$. Fortunately, the real world is more complicated than our simple model. It has been found that, for frequencies greater than or of order 1 Hz, scattering from inhomogeneities in the Earth reduces the coherence length substantially below a wavelength.[3] Thus, in this frequency band, the calculation leading to Eq. (12) does not appear to be an underestimate, but is rather a substantial overestimate.

Now we can use these formulas to produce some numbers. For definiteness and simplicity we will assume that we can ignore the resonance or the damping of the antenna masses down to 1 Hz, and that the antenna baseline is long enough that the short-wavelength approximation is adequate. Then

$$\left[(\omega_0^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2\cong\omega^4|\Delta x(\omega)|^2\,. \quad (19)$$

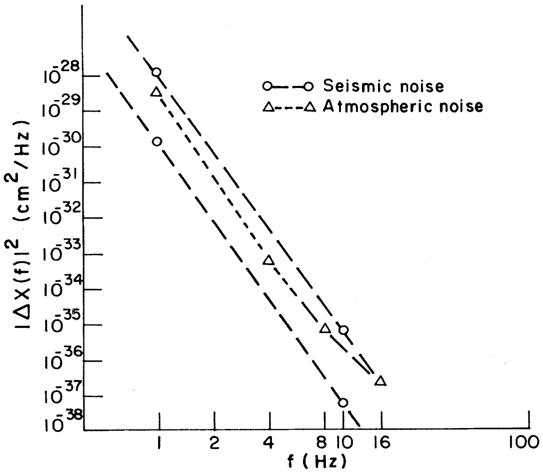

FIG. 2. Interferometer path-length-difference power spectrum, due to random gravitational forces from seismic noise and from atmospheric noise. Two estimates of the seismic noise are shown, corresponding to an average site and a quiet site.

For the air our formula reads

$$|\Delta x(\omega)|^2=\frac{8\pi^3}{3}G^2 v_s^2\frac{\rho_a^2}{p_a^2}\frac{|\Delta p(\omega)|^2}{\omega^6}\,. \quad (20)$$

For the Earth, we will use our formula based on one degree of freedom, since accounting for three degrees of freedom is approximately compensated by the admixture of noncompressional waves with P waves. We have

$$|\Delta x(\omega)|^2=\frac{16\pi^2}{3}\frac{G^2\rho_e^2}{\omega^4}|\Delta X(\omega)|^2\,. \quad (21)$$

Data for the air-pressure power spectrum come from Posmentier[4] and Balachandran.[5] The spectrum can be approximated by $|\Delta p(f)|^2=3\times10^3$ nbar^2/Hz (1 Hz/f)^2 between $f=\frac{1}{2}$ and 3 Hz, and a constant value of 3×10^2 nbar^2/Hz at higher frequencies. For ground motion, the compilation of Fix[6] is used. Between $f=\frac{1}{10}$ and 10 Hz, $|\Delta X(f)|^2$ is proportional to (1 Hz/f).^{4} At an average site, the constant of proportionality is about 3×10^{-13} cm^2/Hz, while at quiet sites the constant may be two orders of magnitude smaller. The calculated path-length-difference power spectra are graphed in Fig. 2. An optimistic but possible value for the intrinsic noise of the interferometer (shot noise in 100 W of laser illumination) is 10^{-34} cm^2/Hz. Depending on the noisiness of the site, observations at this sensitivity will be prevented by random gravitational forces below some frequency in the range of 5 to 10 Hz. Even at a quiet site, sometimes the noise level will be augmented by transient seismic or atmospheric events or by anthropogenic noise.

FORCES DUE TO MOTION OF MASSIVE BODIES

The gravitational force due to moving massive bodies will also affect the interferometer arm lengths. (The effect on a resonant antenna has been calculated by Suzuki and Hirakawa.[7]) Unless the rate of encounters with such objects is large, it is inappropriate to treat these disturbances as a stationary random process, as we could for the sources considered in the previous section. Instead, we analyze the characteristics of individual events.

Consider a point mass M moving at ground level with constant velocity v oriented in the y direction, with impact parameter b relative to a test mass m. (See Fig. 3.) Then, it is easy to show that the x component of the force is

$$\frac{F_x}{m}=\frac{GMb}{r^3(t)}=\frac{GMb}{(b^2+v^2t^2)^{3/2}} \tag{22}$$

and that the y component is

$$\frac{F_y}{m}=\frac{GMvt}{r^3(t)}=\frac{GMvt}{(b^2+v^2t^2)^{3/2}}\ , \tag{23}$$

where $t=0$ is taken to be the time of closest approach. To study the detectability of this signal against a background noise whose power spectrum is known, it is useful to study the Fourier transforms of these expressions. The transforms are tabulated by Erdelyi *et al.*[8] (Note that the x component is even in time, so only the cosine transform contributes, while the y component is odd in time, so only the sine transform contributes.) These transforms are equal to some constants times $wK_0(w)$ (y component) and $wK_1(w)$ (x component), where $w=b\omega/v$ and K_0 and K_1 are the first two modified Bessel functions of the third kind. Near $w=1$ ($\omega=v/b$) the transforms are equal to a number of order unity times $2GM/bv$. For large w, the transforms decline steeply, asymptotically going as $e^{-w}/w^{1/2}$. The equivalent power spectral density is given by the square of the Fourier transform divided by the temporal width of the pulse, $\Delta T=b/v$. This gives

$$\omega^4x^2(\omega)=\frac{1}{\omega}\left[\frac{2GM}{b^2}\right]^2e^{-2w}\ . \tag{24}$$

As an example, consider an airplane with a mass of 100 tons and speed 250 m/sec, passing 10 km away. The characteristic frequency is $\omega=v/b=\frac{1}{35}$ sec^{-1}. The acceleration spectral density at 10 Hz ($w=2100$) is

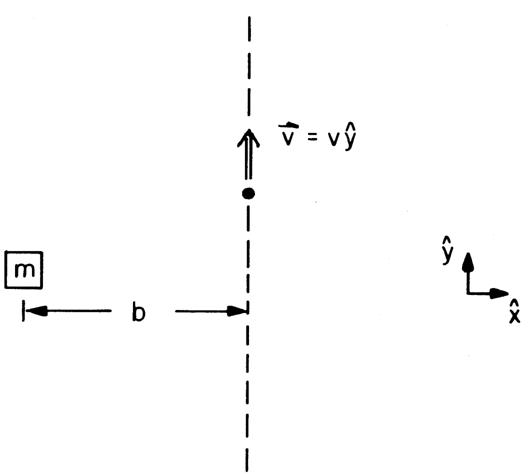

FIG. 3. An encounter of a test mass with a moving massive body.

$e^{-4200}\times3\times10^{-24}$ cm^2/sec^4 Hz, a preposterously small number. A 1-kg rabbit running 100 m away at a speed of 10 m/sec gives $e^{-1260}\times3\times10^{-26}$ cm^2/sec^4 Hz, also minuscule. On the other hand, the same rabbit running within 1 m of the end mass (probably as close as it could get) would have $w=6$. Then the acceleration spectral density is 2×10^{-23} cm^2/sec^4 Hz. The equivalent displacement spectral density is 1×10^{-30} cm^2/Hz, which is not at all negligible compared to the potential sensitivity of gravitational-wave antennas. (Note that we have computed the effect on a single mass only. This is a good approximation for impact parameters b small compared to the baseline L. For larger b, the differential motion is smaller than the motion of a single mass.)

Spero[9] has also considered the problem of gravitational disturbances by moving massive bodies. Instead of the assumption of constant velocity made above, he considers an object of mass M which suddenly starts moving at velocity v, stopping again after an interval t. The net gravitational displacement in a time t of a test mass a distance b away is $\Delta x=Mvt^3/b^3$. The Fourier transform of this pulse has a ω^{-2} dependence on frequency, rather than the exponential dependence found above. This makes a larger class of objects significant sources of random gravitational forces at frequencies large compared to $\omega=v/b$. Of course, the worst case is if an object executes an oscillatory motion. Then the disturbance is concentrated at the oscillation frequency (and perhaps its harmonics).

DISCUSSION

By any ordinary standard, the random gravitational forces at the surface of the Earth are small. Yet, we have found that they set a fundamental limit to the performance of a high-sensitivity gravitational-wave antenna below 10 Hz. This is not to say that achieving performance limited by random gravitational forces will be easy. One measure of the challenge is that the motion X of a point on the Earth's surface at 1 Hz is many orders of magnitude in amplitude larger than the motion x of a test mass induced by random gravitational forces due to seismic density fluctuations. This is easily seen by noting that the end-mass motion due to gravitational perturbations is $x=4(G\rho/\omega^2)X=10^{-8}X$ (at 1 Hz), while (with no isolation) motion due to mechanical coupling is $x=X$. The elastic displacement of the Earth may also be substantially larger than the gravitational displacement of a test mass in the case of a moving body accelerating against the Earth, as in the model of Spero. This means that substantial effort must be put into mechanical vibration-isolation systems in order to attain the highest possible sensitivity. Still, the random gravitational forces set a noise floor below which no terrestrial antenna will see.

APPENDIX: LONG-WAVELENGTH APPROXIMATION

For density fluctuations with $\lambda\gg L$, the random gravitational disturbance to an interferometer is not the incoherent sum of the forces at the individual test masses. Rather, we are dealing with a fluctuating gradient in the gravitational force across the interferometer. It is necessary to explicitly take account of the difference in the gravitational force at the different test masses due to each fluctuating region. (It is not necessary in the short-wavelength case because the random force is strongly dominated by the nearest few cells, as indicated by the strong dependence of the sum over cells on the inner cutoff radius. If the wavelength is short compared to the spacing between the masses, the nearest cells to each mass are different ones, and are by assumption uncorrelated. If the wavelength is long, it is the same cells which are the nearest ones to each test mass.) Thus we are interested in the quantity

$$\left[\frac{F_{0x}}{m}-\frac{F_{1x}}{m}\right]-\left[\frac{F_{0y}}{m}-\frac{F_{2y}}{m}\right],$$

where F_{0x} is the x component of the force on the central mass, F_{2y} is the y component of the force on end mass 2, etc. To first order in L/r (remembering that $r \geq \lambda$), this quantity is equal to

$$\frac{3GML}{r^3}(\sin^2\theta-\cos^2\theta) .$$

The analog to $4\cos^2\theta/r^4$ in the short-wavelength case [see Eq. (5)] is the quantity

$$\frac{9L^2}{r^6}(\sin^4\theta+\cos^4\theta-2\sin^2\theta\cos^2\theta) .$$

In place of Eq. (7) we have

$$\left[(\omega_0{}^2-\omega^2)^2+\frac{\omega^2}{\tau^2}\right]|\Delta x(\omega)|^2 = \frac{336}{15\pi^5}\frac{L^2G^2}{v^6}\omega^6|\Delta M(\omega)|^2 . \qquad (25)$$

The two extra powers of ω mean that, when measured spectra are plugged in for $|\Delta M(\omega)|^2$, the random gravitational force is a less steeply falling function of frequency range discussed in the main body of the paper. Naively evaluating the formulas for the two cases at $\omega=2\pi v/L$ (where the derivations for both are invalid), the long-wavelength formula gives a number 16 times larger (times 4 in amplitude) than the short-wavelength formula. The designer of gravitational-wave antennas will be interested that in this long-wavelength (small-antenna) regime, the gravity gradient noise grows linearly in amplitude with antenna baseline. For a 1-m antenna, the wavelength is equal to the baseline at 300 Hz for sound in air, at higher than 5 kHz for P waves in the Earth. For a 30-m antenna, these frequencies are 10 and 200 Hz, respectively. At an average site, the gravitational noise goes below the shot noise for 100 W illumination (10^{-34} cm^2/Hz) near 2 Hz for a 1-m antenna, by 6 Hz for a 30-m antenna.

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