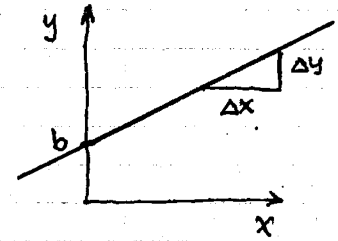
GRAPHS

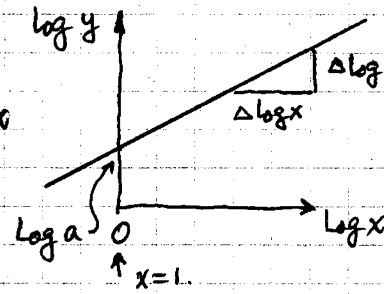
1) Linear relations: $y = mx + b$



$\frac{\Delta y}{\Delta x} = m$ (slope)
positive or negative.

2) Power law; $y = a x^m$

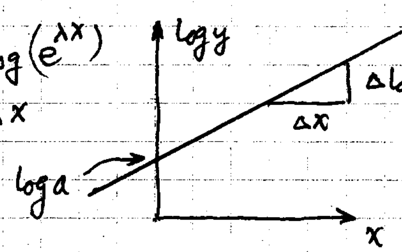
$\log y = \log a + m \log x$



$\frac{\Delta \log y}{\Delta \log x} = m$ (slope)
positive or negative
(log-log plot)

3) Exponential: $y = a e^{\lambda x}$

$$\log y = \log a + \log(e^{\lambda x}) = \log a + \lambda x$$



$\frac{\Delta \log y}{\Delta x} = \lambda$ (slope)
positive or negative
(semi-log plot)

If $\frac{y_2}{y_1} = 2 = \frac{a e^{\lambda x_2}}{a e^{\lambda x_1}} = e^{\lambda(x_2 - x_1)}$, $\log 2 = \lambda(x_2 - x_1) = .69$

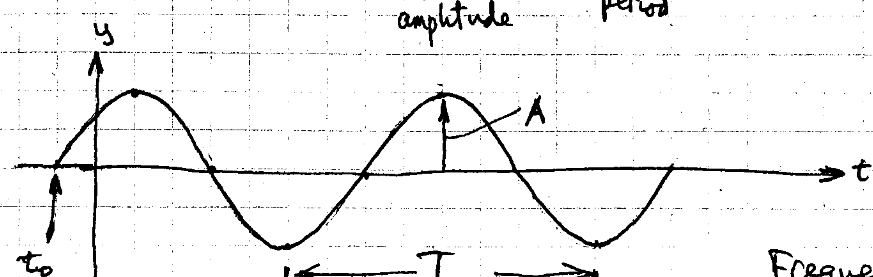
$$x_2 - x_1 = \frac{.69}{\lambda}$$

If x is time, $t_2 - t_1 = \frac{.69}{\lambda}$ = doubling time ($y(t_2) = 2 * y(t_1)$)
if λ is positive

= half-life ($y(t_2) = \frac{1}{2} y(t_1)$)
if λ is negative

4) Harmonic Analysis: $y = A \sin\left(\frac{2\pi}{T}(t - t_0)\right)$

initial phase
amplitude
period



Frequency $f = \frac{1}{T}$

Probability & Statistics

Let the probability that $x_1 \le x \le x_2$ be $\int_{x_1}^{x_2} P(x)\,dx$ ← probability density

Normalization: $1 = \int_{-\infty}^{\infty} P(x)\,dx$

$$\bar{x} = \int_{-\infty}^{\infty} x\,P(x)\,dx \quad \text{(mean of } x\text{)}$$

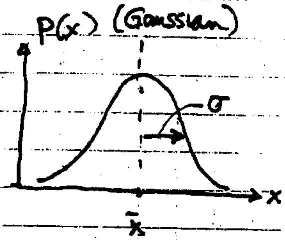
$$\overline{(x-\bar{x})} = \int_{-\infty}^{\infty} (x-\bar{x})\,P(x)\,dx = \bar{x} - \bar{x}\int_{-\infty}^{\infty} P(x)\,dx = \bar{x} - \bar{x} = 0$$

$$\overline{(x-\bar{x})^2} = \int_{-\infty}^{\infty} (x-\bar{x})^2\,P(x)\,dx = \int_{-\infty}^{\infty} (x^2 - 2x\bar{x} + \bar{x}^2)\,P(x)\,dx = \overline{x^2} - 2\bar{x}\bar{x} + \bar{x}^2$$

$$= \overline{x^2} - \bar{x}^2 \quad \text{mean square deviation.}$$

σ (standard deviation) $= \sqrt{\overline{(x-\bar{x})^2}}$ = r.m.s. deviation

root mean square



If x, y are independent of each other $P(x,y) = P_x(x)\,P_y(y)$

Normalization: $1 = \int P(x,y)\,dx\,dy = \int P_x(x)\,dx \cdot \int P_y(y)\,dy = 1\cdot 1$

Let $z = \underbrace{a}x + \underbrace{b}y$ (constants) $\quad \bar{z} = \int P(x,y)\,z\,dx\,dy = \int P_x(x)\,P_y(y)\,(ax+by)\,dx\,dy$

$$= a\int P_x(x)\,x\,dx \int P_y(y)\,dy + b\int P_x(x)\,dx \int P_y(y)\,y\,dy = a\bar{x} + b\bar{y}$$

$$\overline{(z-\bar{z})^2} = \overline{z^2} - \bar{z}^2 = \int (ax+by)^2\,\underbrace{P(x,y)}_{P_x(x)P_y(y)}\,dx\,dy - a^2\bar{x}^2 - b^2\bar{y}^2 - 2ab\,\bar{x}\bar{y}$$

$$= a^2\overline{x^2} + b^2\overline{y^2} + 2ab\,\bar{x}\bar{y} - a^2\bar{x}^2 - b^2\bar{y}^2 - 2ab\,\bar{x}\bar{y}$$

$$= a^2(\overline{x^2} - \bar{x}^2) + b^2(\overline{y^2} - \bar{y}^2) = a^2\sigma_x^2 + b^2\sigma_y^2 = \sigma_z^2$$

Suppose $z = \frac{1}{n}x + \frac{1}{n}x + \dots + \frac{1}{n}x$ where the x's are n independent samples from the same population

Then $\bar{z} = n\cdot\frac{1}{n}\bar{x} = \bar{x}$

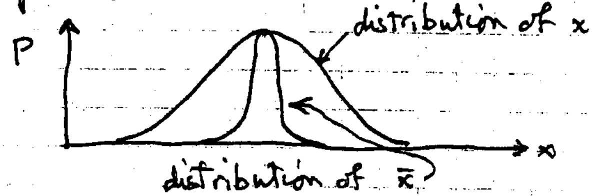
$$\overline{(z-\bar{z})^2} = n\cdot\frac{1}{n^2}\sigma_x^2 = \frac{\sigma_x^2}{n}$$

$$\sigma_z = \frac{\sigma_x}{\sqrt{n}}$$

← std. dev. of original population

← This is why the mean of a sample is a more reliable measure of the mean of the population than an individual member of the population.

std. dev. of a population consisting of the mean of a sample of n members



More generally, suppose $z = f(x,y)$.

Given $\bar{x}, \bar{y}, \sigma_x, \sigma_y$, find $\bar{z}, \sigma_z$

Assume $\sigma_x \ll \bar{x}$, $\sigma_y \ll \bar{y}$ so that x stays near $\bar{x}$, y stays near $\bar{y}$

Do a Taylor expansion of $f(x,y)$ around $x = \bar{x}$, $y = \bar{y}$:

$$z = f(\bar{x},\bar{y}) + (x-\bar{x})\frac{\partial f}{\partial x}(\bar{x},\bar{y}) + (y-\bar{y})\frac{\partial f}{\partial y}(\bar{x},\bar{y}) + \dots$$

$$\bar{z} \approx \underbrace{\overline{f(\bar{x},\bar{y})}}_{f(\bar{x},\bar{y})} + \underbrace{\overline{(x-\bar{x})}}_{0}\frac{\partial f}{\partial x}(\bar{x},\bar{y}) + \underbrace{\overline{(y-\bar{y})}}_{0}\frac{\partial f}{\partial y}(\bar{x},\bar{y}) \approx f(\bar{x},\bar{y}) = \bar{z}$$

Then $z - \bar{z} = (x-\bar{x})\frac{\partial f}{\partial x}(\bar{x},\bar{y}) + (y-\bar{y})\frac{\partial f}{\partial y}(\bar{x},\bar{y})$

$$\sigma_z^2 = \overline{(z-\bar{z})^2} = \overline{(x-\bar{x})^2}\left[\frac{\partial f}{\partial x}(\bar{x},\bar{y})\right]^2 + \overline{(y-\bar{y})^2}\left[\frac{\partial f}{\partial y}(\bar{x},\bar{y})\right]^2$$ if $x-\bar{x}$, $y-\bar{y}$ are uncorrelated.

$$\sigma_z^2 = \left[\frac{\partial f}{\partial x}(\bar{x},\bar{y})\right]^2\sigma_x^2 + \left[\frac{\partial f}{\partial y}(\bar{x},\bar{y})\right]^2\sigma_y^2$$

Example: $F = ma$. Relate $\bar{F}, \sigma_F$ to $\bar{m}, \sigma_m, \bar{y}, \sigma_m$

$$\bar{F} = \bar{m}\,\bar{a}$$

$$\frac{\partial F}{\partial m}(\bar{m},\bar{a}) = \bar{a}, \quad \frac{\partial F}{\partial a}(\bar{m},\bar{a}) = \bar{m}$$

$$\sigma_F^2 = \left[\frac{\partial F}{\partial m}(\bar{m},\bar{a})\right]^2\sigma_m^2 + \left[\frac{\partial F}{\partial a}(\bar{m},\bar{a})\right]\sigma_a^2 = \bar{a}^2\sigma_m^2 + \bar{m}^2\sigma_a^2$$

$$\left(\frac{\sigma_F}{\bar{F}}\right)^2 = \left(\frac{\sigma_m}{\bar{m}}\right)^2 + \left(\frac{\sigma_a}{\bar{a}}\right)^2$$

Populations are usually characterized by two numbers:

a mean value $\bar{x}$ (usually the arithmetic mean, $\frac{\sum_{i=1}^{n} x_i}{n}$

and a standard deviation σ

$$\sigma \equiv \sqrt{\overline{(x-\bar{x})^2}}$$

(also called the r.m.s (root-mean-square) deviation)

In general, $\bar{x}$ and σ are independent parameters. However, for one particular distribution, The Poisson distribution

$$P(n) = e^{-\mu} \frac{\mu^n}{n!} \qquad (n = 0, 1, 2, \ldots$$

it is easy to show that $\bar{n} = \sum_n n P(n) = \mu$

$$\sigma = \sqrt{\overline{(n-\bar{n})^2}} = \sqrt{\overline{(n-\mu)^2}} = \sqrt{\mu} = \sqrt{\bar{n}}$$

Thus for a Poisson distribution, both the mean and standard deviation are determined by the single parameter μ.

Application to counting statics: If the probability of a count occuring in infinitesimal time dt is $\lambda\, dt$, then the probability of n counts occurring in time T is $P(n) = e^{-\mu} \frac{\mu^n}{n!}$, with $\mu = \lambda T$.

Since $\sigma = \sqrt{\bar{n}}$, $\frac{\sigma}{\bar{n}}$ (fractional uncertainty) $= \frac{\sqrt{\bar{n}}}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$

Example: $\bar{n} = 10{,}000$ counts, $\sigma = 100$ counts, $\frac{\sigma}{\bar{n}} = \frac{1}{100} = 1\%$

Example: For reaction A, $N_A = 1000$ counts are detected in a certain time
" " B, $N_B = 600$ " " " the same "

What is the fractional uncertainty of $N_A - N_B = \Delta$

$$\sigma_\Delta^2 = \sigma_A^2 + \sigma_B^2 = 1000 + 600 = 1600, \quad \sigma_\Delta = 40, \quad \bar{\Delta} = 1000 - 600 = 400$$

$$\frac{\sigma_\Delta}{\bar{\Delta}} = \frac{40}{400} = \frac{1}{10} = 10\%$$

Example: 5 independent measurements of a quantity yields a mean of 40, and a standard deviation of 4.0 ← for the population

How many <u>additional</u> measurements needed to determine the mean to 2%.

We need n such that $\frac{\left(\frac{4}{\sqrt{n}}\right)}{40}$ = .02

$\left(\frac{4}{\sqrt{n}}\right)$ ← standard deviation of mean of n.

40 ← population mean

$$\frac{4}{\sqrt{n}} = .02 \times 40 = .8, \quad \frac{4}{.8} = 5 = \sqrt{n}$$

$n = 25$ measurements are need.
5 measurements have already been made.
20 <u>add</u>itional measurements are needed.